

# GAUSSIAN FLUCTUATIONS OF JACK-DEFORMED RANDOM YOUNG DIAGRAMS

MACIEJ DOŁĘGA AND PIOTR ŚNIADY

**ABSTRACT.** We introduce a large class of growing random Young diagrams related to the notion of *Jack characters*. We show that each such a random Young diagram converges asymptotically to some limit shape and that the fluctuations around the limit are asymptotically Gaussian.

*Random partitions* (or *random Young diagrams*) occur in mathematics and physics in a wide variety of contexts, in particular in the Gromov–Witten and Seiberg–Witten theories, see the overview articles of Okounkov [Oko03] and Vershik [Ver95] for an introduction to the subject and even more motivations for studying this subject.

One of our main personal motivations is the link to the *random matrix theory*: certain random Young diagrams can be regarded as discrete counterparts of some interesting ensembles of random matrices.

In the current paper we shall explore this link on a particular example of random Young diagrams related to a celebrated class of random matrices called  $\beta$ -ensembles or  $\beta$ -log gases [For10]. Recall that  $\beta$ -ensembles are the probability distributions on  $\mathbb{R}^n$  with the density of the form

$$p(x_1, \dots, x_n) = \frac{1}{Z} e^{V(x_1) + \dots + V(x_n)} \prod_{i < j} |x_i - x_j|^\beta,$$

where  $V$  is some real-valued function and  $Z$  is the normalization constant. In the special cases  $\beta \in \{1, 2, 4\}$  they describe the joint distribution of the eigenvalues of random matrices with natural symmetries; the investigation of such ensembles for a generic value of  $\beta$  is motivated, among others, by statistical mechanics.

Opposite to the special cases  $\beta \in \{1, 2, 4\}$ , in the generic case of  $\beta$ -ensembles there seems to be no obvious unique way of defining their discrete counterpart and several alternative approaches are available, see the

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work of Moll [Mol15] as well as the work of Borodin, Gorin and Guionnet [BGG16].

In the current paper we took a different approach which has its roots in the representation theory of the symmetric groups. Indeed, for a given reducible representation  $\rho$  of the symmetric group  $\mathfrak{S}(n)$  one can ask about the statistical properties of the random irreducible component of  $\rho$ ; the corresponding random Young diagram can be regarded as a discrete counterpart of some  $\beta$ -ensemble for the special case  $\beta = 2$ . In order to generalize this concept to a generic value of  $\beta > 0$  we replace the character theory of the symmetric groups by its deformation related to *Jack polynomials*.

An additional motivation for considering this particular class of random Young diagrams stems the research related to the problem of finding extremal characters of the infinite symmetric group  $\mathfrak{S}(\infty)$ , solved by Thoma [Tho64]. Vershik and Kerov [VK81] found an alternative, more conceptual proof of Thoma's result, which was based on the observation that *characters of  $\mathfrak{S}(\infty)$  are in a natural bijective correspondence with certain sequences  $(\lambda_1 \nearrow \lambda_2 \nearrow \dots)$  of growing random Young diagrams*.

The original Thoma's problem can be equivalently formulated as finding all homomorphisms from the ring of symmetric functions to real numbers which are *Schur-positive*, i.e. which take non-negative values on all Schur polynomials. In this formulation the problem naturally asks for generalizations in which Schur polynomials are replaced by another interesting family of symmetric functions. Kerov, Okounkov and Olshanski [KOO98] considered the particular case of *Jack polynomials* and they proved that a direct analogue of Thoma's result holds true also in this case. The main idea behind their proof was that the probabilistic viewpoint from the above-mentioned work of Vershik and Kerov [VK81] can be adapted to the new setting of Jack polynomials. Thus, a side product of the work of Kerov, Okounkov and Olshanski is an interesting, natural class of random Young diagrams. The Young diagrams from this class fit very nicely into the framework which we consider in the current paper, see Section 1.13.

There are also some other motivations for studying the class of random Young diagrams considered in the current paper; motivations which come from the algebraic combinatorics. The main technical tool which we shall use in the proofs is a particularly nice relationship between two different multiplicative structures on the *algebra of  $\alpha$ -polynomial functions*. Despite some progress in this topic, the details of this relationship remain elusive and several algebraic-combinatorial conjectures [Śni16, Conjectures 0.1 and 0.6] which indicate existence of some hidden rich underlying structure remain open. These conjectures might be seen as a heuristic indication

that the probability measures which we investigate in the current paper are defined *in the right way*.

## 1. INTRODUCTION

The Reader should be advised that, in order to keep this introduction fast and lightweight, we decided to postpone the overview of basic notations related to partitions and permutations to Section 1.6.

**1.1. The main problem.** In the current paper we shall address the following quite general problem.

*Problem 1.1* (The main problem). Suppose that for each  $n \geq 1$  a probability measure  $\mathbb{P}_n$  on the set  $\mathbb{Y}_n$  of Young diagrams with  $n$  boxes is given implicitly in terms of the *character*  $\chi_n$ . What can we say about the statistical properties of the corresponding sequence  $(\lambda_n)$  of random Young diagrams? We are interested in the asymptotic setup  $n \rightarrow \infty$  in which the number of boxes tends to infinity.

The above problem might sound somewhat cryptic; indeed, we have not explained what a *character* is. We shall use this term in the following two contexts:

- the word *character* might refer to the *character of the symmetric group*  $\mathfrak{S}(n)$ ,
- or, more generally and more importantly for the purposes of the current paper, the word *character* might refer to the *Jack character* which is a one-parameter deformation of the characters of the symmetric groups; a deformation which is related to *Jack polynomials* and  $\beta$ -ensembles.

Our main Problem 1.1 is classical and quite well understood in the first of these two contexts: the characters of the symmetric groups [Bia01, Śni06a]. The main result of the current paper is an extension of these results to the latter context of Jack characters and Jack polynomials.

**1.2. Random Young diagrams and characters.** In either of the above two contexts, for a given integer  $n \geq 1$  we start with a family of *irreducible characters*  $(\chi_\lambda)$  indexed by a Young diagram  $\lambda$  with  $n$  boxes. Each irreducible character  $\chi_\lambda: \mathcal{P}_n \rightarrow \mathbb{R}$  is supposed to be a function on the set of partitions of the number  $n$  with the property that

$$\chi_\lambda(1^n) = 1,$$

where  $1^n = (1, 1, \dots, 1) \vdash n$  is a partition of  $n$  which consists of  $n$  parts, each equal to 1. We also assume that the irreducible characters are linearly independent.

The elements of the convex hull of the irreducible characters  $(\chi_\lambda : \lambda \in \mathbb{Y}_n)$  will be called *reducible characters*. There is a simple bijective correspondence between the set of reducible characters and the set of probability measures on the set  $\mathbb{Y}_n$  of Young diagram with  $n$  boxes given as follows: to a given character  $\chi$  we associate the probability measure  $\mathbb{P}_\chi$  which gives the coefficients in the expansion of  $\chi$  in the basis of irreducible characters

$$(1.1) \quad \chi = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{P}_\chi(\lambda) \chi_\lambda.$$

Thus, our main Problem 1.1 is essentially a question about the relationship between the statistical properties of the random Young diagram distributed according to the above probability measure  $\mathbb{P}_\chi$  and the properties of the corresponding reducible character  $\chi$ . Note, however, that Problem 1.1 depends implicitly on the the initial choice of the family  $(\chi_\lambda)$  of the irreducible characters. We shall discuss some interesting choices of this family in the following.

### 1.3. Characters of the symmetric groups and random Young diagrams.

Let us start with the classical choice of the characters of the symmetric groups. Impatient readers may fast forward to Section 1.4 where we shall discuss the more general concept of the Jack characters which is in the focus of the current paper.

**1.3.1. Irreducible characters.** For a Young diagram  $\lambda \in \mathbb{Y}_n$  with  $n$  boxes let  $\rho_\lambda : \mathfrak{S}(n) \rightarrow M_k(\mathbb{R})$  denote the corresponding *irreducible representation* [Sag01] of the symmetric group  $\mathfrak{S}(n)$ . For a permutation  $\pi \in \mathfrak{S}(n)$  we define the value of the *irreducible character*  $\chi_\lambda$  as the fraction

$$(1.2) \quad \chi_\lambda(\pi) := \frac{\text{Tr } \rho_\lambda(\pi)}{\text{Tr } \rho_\lambda(\text{id})}.$$

Note that in the literature one usually uses the term *irreducible character* for the numerator of the above ratio.

Since one can identify a permutation  $\pi$  with its cycle decomposition, it follows that the irreducible character  $\chi_\lambda(\pi)$  is also well-defined if  $\pi \in \mathcal{P}_n$  is a partition of  $n$ . In this way we obtain an interesting family  $(\chi_\lambda)$  of functions on  $\mathcal{P}_n$  for which we can consider a concrete version of our Problem 1.1.

**1.3.2. Reducible representations and reducible characters.** The most natural setup in which some interesting reducible characters arise is the one in which we start with some (possibly reducible) representation  $\rho : \mathfrak{S}(n) \rightarrow$

$M_k(\mathbb{R})$  of the symmetric group. Such a reducible representation can be always decomposed into irreducible components:

$$\rho = \bigoplus_{\lambda \in \mathbb{Y}_n} m_\lambda \rho_\lambda,$$

where  $m_\lambda \in \{0, 1, 2, \dots\}$  is the multiplicity of the irreducible representation  $\rho_\lambda$ .

In this setup it is natural to ask about the statistical properties of a *random irreducible component* of the representation  $\rho$ . To be more specific, to a given Young diagram  $\lambda$  we associate a probability which is proportional to the total dimension of all isotypic irreducible components of  $\rho$  equivalent to  $\rho_\lambda$ :

$$(1.3) \quad \mathbb{P}_\rho(\lambda) := \frac{m_\lambda \cdot \dim \rho_\lambda}{\dim \rho}.$$

The representation  $\rho$  gives rise to the corresponding reducible character defined analogously to (1.2) as the fraction

$$\chi_\rho(\pi) := \frac{\text{Tr } \rho_\rho(\pi)}{\text{Tr } \rho_\rho(\text{id})},$$

where just as before  $\pi$  can be regarded either as a permutation from  $\mathfrak{S}(n)$  or as a partition from  $\mathcal{P}_n$ .

From our perspective it is very interesting that the probability measure  $\mathbb{P}_\rho$  defined by (1.3) is equal to the probability measure  $\mathbb{P}_{\chi_\rho}$  which is assigned via the general prescription (1.1) to the reducible character  $\chi := \chi_\rho$ .

*Example 1.2.* Consider the left-regular representation  $\rho$  of the symmetric group  $\mathfrak{S}(n)$  given by the left action of  $\mathfrak{S}(n)$  on its group algebra  $\mathbb{R}[\mathfrak{S}(n)]$ . In this case the character

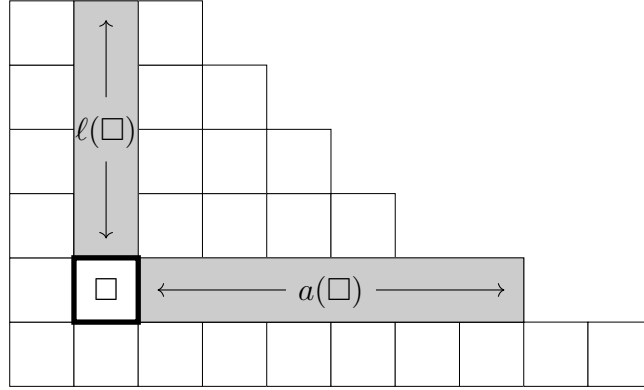
$$(1.4) \quad \chi_\rho(\pi) = \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases}$$

takes a particularly simple form. The corresponding probability measure (1.3) is the celebrated *Plancherel measure*

$$(1.5) \quad \mathbb{P}(\lambda) := \frac{(\dim \rho_\lambda)^2}{n!}$$

which has been a subject of several investigations, see [LS77, VK77, Ker93a] to mention just a few.

The celebrated *hook-length formula*, discovered by Frame, Robinson and Thrall [FRT54] allows computation of the dimension  $\dim \rho_\lambda$  in an elegant, combinatorial way. In order to explain it, we need some basic notations related to Young diagrams. We use the Cartesian convention for numbering boxes of the Young diagrams; this means that  $(i, j) \in \lambda$  is the box lying

FIGURE 1. Arm and leg length of a box  $\square$  of a Young diagram.

in the  $i$ -th column and  $j$ -th row of a Young diagram  $\lambda$ . For a given box  $\square := (i, j) \in \lambda$  of a given Young diagram  $\lambda$  we will use the following two quantities (cf. [Mac95, Chapter I]): the *arm-length* of  $\square$  defined as  $a(\square) := \lambda_j - i$  which gives the number of boxes lying strictly to the right of  $\square$ , and its *leg-length* defined by  $\ell(\square) := \lambda_i^t - j$  which gives the number of boxes lying strictly above  $\square$ , see Figure 1.

Then, the hook-length formula states that

$$\dim \rho_\lambda = \frac{n!}{\prod_{\square \in \lambda} h(\square)},$$

where

$$(1.6) \quad h(\square) := a(\square) + \ell(\square) + 1$$

is the *hook-length* of  $\square \in \lambda$ .

Thus the Plancherel measure (1.5) can be equivalently written in the form

$$(1.7) \quad \mathbb{P}(\lambda) := \frac{n!}{\prod_{\square \in \lambda} (h(\square))^2}.$$

**1.4. Jack characters.** It is the time to present the second context, the more general one, of the two contexts which we announced in Section 1.1. In this context for the family of irreducible characters  $(\chi_\lambda)$  we take *irreducible Jack characters*.

**1.4.1. Irreducible Jack characters.** Our starting point is the family of *Jack polynomials*  $J_\lambda^{(\alpha)}$  [Jac71] which can be regarded as a deformation of the family of Schur polynomials; a deformation that depends on the parameter  $\alpha$ . We use the normalization of Jack polynomials from [Mac95, Section VI,10].

In the following we fix the value of the deformation parameter  $\alpha > 0$ . We expand Jack polynomial in the basis of power-sum symmetric functions:

$$(1.8) \quad J_\lambda^{(\alpha)} = \sum_{\pi} \theta_\pi^{(\alpha)}(\lambda) p_\pi.$$

The above sum runs over partitions  $\pi$  such that  $|\pi| = |\lambda|$ . For a given integer  $n \geq 1$ , any Young diagram  $\lambda \in \mathbb{Y}_n$  and any partition  $\pi \in \mathcal{P}_n$  we define the value of the *irreducible Jack character*  $\chi_\lambda^{(\alpha)}$  as

$$(1.9) \quad \chi_\lambda^{(\alpha)}(\pi) := \alpha^{-\frac{\|\pi\|}{2}} \frac{z_\pi}{n!} \theta_\pi^{(\alpha)}(\lambda),$$

where  $\|\pi\| := |\pi| - \ell(\pi)$ .

It is worth pointing out that in the special case  $\alpha = 1$  the corresponding Jack character  $\chi_\lambda^{(1)}(\pi) = \chi_\lambda(\pi)$  coincides with the irreducible character (1.2) of the symmetric group  $\mathfrak{S}(n)$ , see [Las09, DF16].

*Remark 1.3.* The Reader may wonder for the motivations for such a bizarre choice of the normalization in the definition (1.9), in particular for the choice of the exponent of the deformation parameter  $\alpha$ . One possible motivation is *a posteriori*: as we shall see later, with this choice the prescription (1.1) will produce several natural and interesting examples of probability measures on the set of partitions. For a good *a priori* motivation we have to wait until Remark 2.2 where we will discuss the intrinsic merits of this choice of the normalization as well as an abstract definition of irreducible Jack characters which does not refer to the notion of Jack polynomials.

**1.4.2. Reducible Jack characters.** We review the general scheme presented in Section 1.2 in the specific context in which for the family of the irreducible characters  $(\chi_\lambda)$  we take the irreducible Jack characters  $(\chi_\lambda^{(\alpha)})$ . In order to emphasise this specific choice, the elements of the convex hull of the irreducible Jack characters will be called *reducible Jack characters*.

One of the prescriptions for finding *interesting* examples of reducible Jack characters is to start with some *interesting* reducible character  $\chi$  of the symmetric group  $\mathfrak{S}(n)$  and to hope that the coefficients in the expansion

$$(1.10) \quad \chi = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{P}_\chi(\lambda) \chi_\lambda^{(\alpha)}$$

in the basis of irreducible Jack characters define a probability measure, as it is a priori not clear that  $\mathbb{P}_\chi(\lambda) \geq 0$ .

The following example fits into this general scheme.

*Example 1.4.* We revisit Example 1.2; let  $\chi := \chi_\rho$  be given by (1.4). The corresponding probability measure on  $\mathbb{Y}_n$  is called the *Jack–Plancherel measure*; Stanley [Sta89] found an explicit expression for it:

$$(1.11) \quad \mathbb{P}(\lambda) := \frac{n!}{\prod_{\square \in \lambda} h_\alpha(\square) h'_\alpha(\square)}.$$

The above formula involves two  $\alpha$ -deformations of the hook-length (1.6), given by

$$\begin{aligned} h_\alpha(\square) &:= Aa(\square) + A^{-1}\ell(\square) + A \\ h'_\alpha(\square) &:= Aa(\square) + A^{-1}\ell(\square) + A^{-1}, \end{aligned}$$

where  $A := \sqrt{\alpha}$ . It is straightforward both from the definition, as well as by comparing the formula (1.11) to (1.7), that the Jack–Plancherel measure coincides with the Plancherel measure for  $\alpha = 1$  and might be considered as its one-parameter deformation. Since there seems to be no representation theory around, no analogue of (1.5) is available, which makes investigation of this probability measure more difficult and requires new methods. The Jack–Plancherel measure has been subject of several investigations, see [Ful04, Mat08, DF16, BGG16, Mol15] among many others.

By comparing (1.11) to (1.7) we see an instance of some informal phenomenon: “*if you want to deform things in a Jack way, you should draw your Young diagram anisotropically, as on Figure 2*”; see Section 1.5.1 for more details.

### 1.5. Conclusion of the main theorem. Convergence of Young diagrams.

The main result of the current paper is somewhat technically involved. We postpone the review of the *hypothesis* of this result until Section 1.7 and we start with a presentation of the *conclusion* part.

**Theorem 1.5** (The main result, informal formulation). *Let  $\alpha > 0$  be fixed; for each  $n$  let  $\chi_n: \mathcal{P}_n \rightarrow \mathbb{R}$  be a reducible Jack character. We impose that the sequence  $(\chi_n)$  fulfills some technical assumptions about its asymptotic behavior; we will specify their details later in Section 1.7.*

*Let  $\lambda_n$  be a random Young diagram with the probability distribution  $\mathbb{P}_{\chi_n}$  associated with  $\chi := \chi_n$  via (1.10). Then the sequence  $(\lambda_n)$  of Young diagrams converges to some limit shape in the limit  $n \rightarrow \infty$  when the number of the boxes tends to infinity.*

*Furthermore, the fluctuations of  $\lambda_n$  around the limit shape are asymptotically Gaussian.*

Impatient readers who are not satisfied with such informal formulations may fast forward to Theorems 1.6, 1.7 and 1.9 where the main result of the paper is stated with more details.



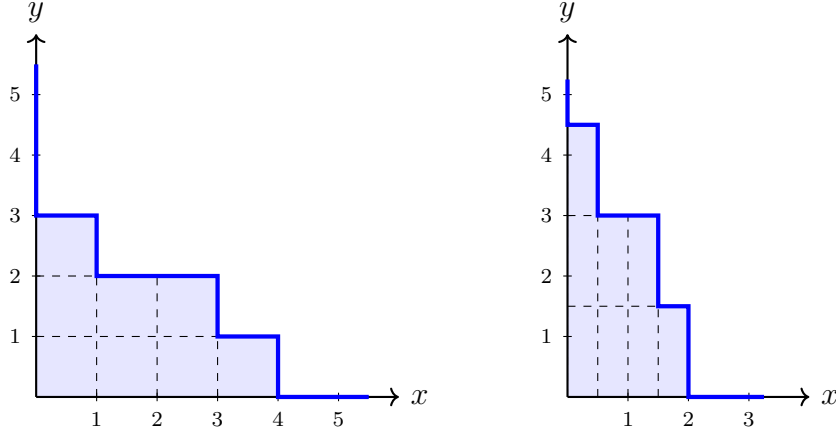


FIGURE 2. A Young diagram  $\lambda = (4, 3, 1)$  shown in the French convention (left) and a generalized Young diagram  $T_{\frac{1}{2}, \frac{3}{2}} \lambda$  (right) obtained by an anisotropic scaling. The dashed lines indicate individual boxes.

The remaining part of this section is devoted to the explanation of the sense in which the convergence of the shapes of the Young diagrams in Theorem 1.5 is to be understood.

**1.5.1. Anisotropic Young diagrams.** The usual way of drawing Young diagrams is to represent each individual box as a unit square, see Figure 2 (left). However, when dealing with random Young diagrams related to Jack polynomials  $J^{(\alpha)}$ , it is more convenient to represent each box as a rectangle with width  $w > 0$  and height  $h > 0$  such that

$$(1.12) \quad \frac{w}{h} = \alpha.$$

A Young diagram viewed like this becomes a polygon contained in the upper-right quarterplane which will be denoted by  $T_{w,h} \lambda$ , see Figure 2 (right). We will refer to such polygons as *anisotropic Young diagrams*; they have been first considered by Kerov [Ker00], who introduced them to prove certain combinatorial identities involving Jack symmetric functions by analytic techniques.

For many asymptotic problems it is convenient to choose  $w$  and  $h$  in such a way that any Young diagram  $\lambda \in \mathbb{Y}_n$  with  $n$  boxes is represented by an anisotropic Young diagram  $T_{w,h} \lambda$  with the unit area; in other words it is desirable to have

$$(1.13) \quad nwh = 1.$$

Equations (1.12) and (1.13) motivate the choice

$$(1.14) \quad w = \sqrt{\frac{\alpha}{n}}, \quad h = \sqrt{\frac{1}{\alpha n}}$$

which will be often used in the following.

We are now able to state a part of Theorem 1.5 a bit more precisely: when we say that *the sequence  $(\lambda_n)$  of Young diagrams converges to some limit shape* we really mean that *the sequence of anisotropic Young diagrams*

$$(1.15) \quad \Lambda_n := T_{\sqrt{\frac{\alpha}{n}}, \sqrt{\frac{1}{\alpha n}}} \lambda_n$$

*converges to some limit shape for  $n \rightarrow \infty$ .* It remains to specify the meaning of the latter convergence.

**1.5.2. Russian convention. Profile of a Young diagram.** We draw (anisotropic) Young diagrams on the plane with the usual Cartesian coordinates  $(x, y)$ . However, it is also convenient to use the *Russian* coordinate system  $(u, v)$  given by

$$u = x - y, \quad v = x + y.$$

This new coordinate system gives rise to the *Russian convention* for drawing (anisotropic) Young diagrams, see Figure 3.

The boundary of a Young diagram  $\lambda$  drawn in the Russian convention (the solid zigzag line on the right-hand side of Figure 3) is a graph of a function  $\omega_\lambda$  which will be called the *profile* of  $\lambda$ . If the Young diagram is replaced by an anisotropic Young diagram  $T_{w,h}\lambda$  we define in an analogous way its profile  $\omega_{T_{w,h}\lambda}$ .

**1.5.3. Law of Large Numbers.** We are now ready to state a part of Theorem 1.5 even more precisely.

**Theorem 1.6** (Law of large numbers). *Let  $(\lambda_n)$  be a sequence of random Young diagrams as in Theorem 1.5. We assume that the corresponding sequence  $(\chi_n)$  of reducible characters fulfills some technical condition about its asymptotic behavior which will be presented in details in Section 2.8.2 as Hypothesis 2.5.*

*Then there exists some deterministic function  $\omega_{\Lambda_\infty}: \mathbb{R} \rightarrow \mathbb{R}$  with the property that*

$$(1.16) \quad \lim_{n \rightarrow \infty} \omega_{\Lambda_n} = \omega_{\Lambda_\infty},$$

*where  $\Lambda_n$  has been defined in (1.15) and the convergence holds true with respect to the supremum norm, in probability. In other words, for each  $\epsilon > 0$*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\|\omega_{\Lambda_n} - \omega_{\Lambda_\infty}\|_\infty > \epsilon) = 0.$$

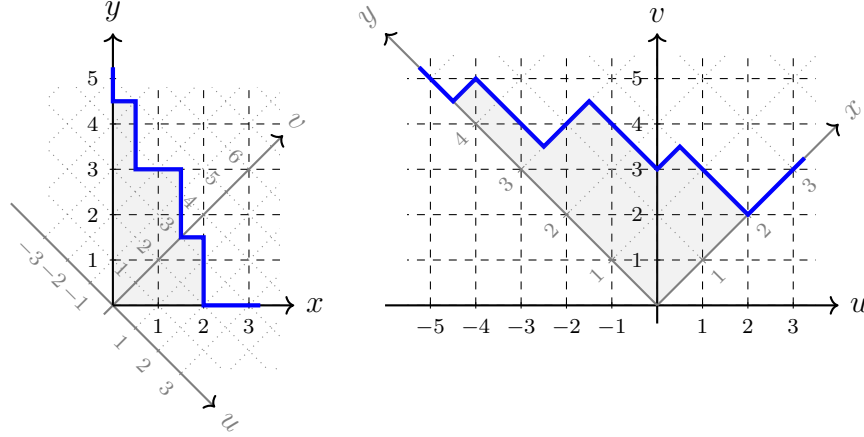


FIGURE 3. The anisotropic Young diagram from Figure 2 shown in the French and Russian conventions. The solid line represents the *profile* of the Young diagram. The coordinates system  $(u, v)$  corresponding to the Russian convention and the coordinate system  $(x, y)$  corresponding to the French convention are shown.

The proof and the details of the hypothesis are postponed to Section 2.8.2. The above function  $\omega_{\Lambda_\infty}$  can be informally regarded as a profile of some mythical “generalized” or “continuous” Young diagram  $\Lambda_\infty$ , which is not random.

1.5.4. *Functionals of shape.* The shape of a profile  $\omega$  can be quantified with the help of the *functionals of shape* defined by

$$(1.17) \quad \mathcal{S}_k^{(1)}(\omega) := (k-1) \int_{-\infty}^{\infty} u^{k-2} \frac{\omega(u) - |u|}{2} du$$

for  $k \in \{2, 3, \dots\}$ .

If  $\omega = \omega_{T_{w,h}\lambda}$  is a profile of some specific anisotropic Young diagram we use the simplified notation

$$\mathcal{S}_k^{(1)}(T_{w,h}\lambda) := \mathcal{S}_k^{(1)}(\omega_{T_{w,h}\lambda}).$$

1.5.5. *Central Limit Theorem, the simpler version.* We are now ready to state precisely the exact formulation of the Central Limit Theorem (CLT) part of the conclusion of the main result (Theorem 1.5).

We encode the shape of an anisotropic Young diagram  $\Lambda_n$  by its profile  $\omega_{\Lambda_n}$ . The difference

$$(1.18) \quad \overline{\Delta}_n(u) := \sqrt{n}(\omega_{\Lambda_n}(u) - \mathbb{E}\omega_{\Lambda_n}(u))$$

is a random function on the real line which quantifies the (suitably rescaled) discrepancy between the shape of the random (anisotropic) Young diagram  $\Lambda_n$  and the ‘average’ shape of  $\Lambda_n$ . We will regard  $\overline{\Delta}_n$  as a Schwartz distribution on the real line  $\mathbb{R}$  or, more precisely, as a *random vector* from this space.

**Theorem 1.7** (Central Limit Theorem, the simpler version). *Let  $(\lambda_n)$  be a sequence of random Young diagrams as in Theorem 1.5. We assume that the corresponding sequence  $(\chi_n)$  of reducible characters fulfills some technical condition about its asymptotic behavior (‘approximate factorization property’, see Definition 1.10 later on).*

*Then the centered random vector  $\overline{\Delta}_n$  converges in distribution to some Gaussian random vector as  $n \rightarrow \infty$ .*

*Remark 1.8.* The space of Schwartz distributions is a dual space; for this reason the above statement about the random vector  $\overline{\Delta}_n$  should be understood in a rather specific sense, formulated with help of some suitable test functions. In particular, in order to prove Theorem 1.7 it is enough to show that the joint distribution of the family of centered random variables

$(X_k^{[n]})_{k \geq 2}$  converges, as  $n \rightarrow \infty$ , to a Gaussian distribution, where

$$(1.19) \quad X_k = X_k^{[n]} := \frac{k-1}{2} \int u^{k-2} \overline{\Delta}_n(u) \, du = \sqrt{n} \left( \mathcal{S}_k^{(1)}(\Lambda_n) - \mathbb{E} \mathcal{S}_k^{(1)}(\Lambda_n) \right)$$

is (up to a simple scalar factor) the value of the Schwartz distribution  $\overline{\Delta}_n$  evaluated on a suitable polynomial test function. In order to keep the notation light, we often make the dependence of  $X_k^{[n]}$  on  $n$  implicit and we write shortly  $X_k$ .

The proof is postponed to Section 2.7.2.

1.5.6. *Example.* For simplicity, let  $\alpha > 0$  be a fixed positive integer. For a given integer  $i > 0$  consider the rectangular Young diagram

$$(i^{\alpha i}) := (\underbrace{i, \dots, i}_{\alpha i \text{ times}})$$

with  $n' := \alpha i^2$  boxes. We will assume that  $n'$  is an even number. A special case  $\alpha = 1$  was considered already by Biane [Bia98, Figures 1–3].

Using a random iterative procedure which is an inverse of the Plancherel growth process introduced by Kerov [Ker96] and which will be presented in detail in a forthcoming paper [DŚ17] (see also Section 1.13) we remove exactly half of the boxes from the rectangular Young diagram  $(i^{\alpha i})$ ; the resulting random Young diagram with  $n := \frac{1}{2}n'$  boxes will be denoted by

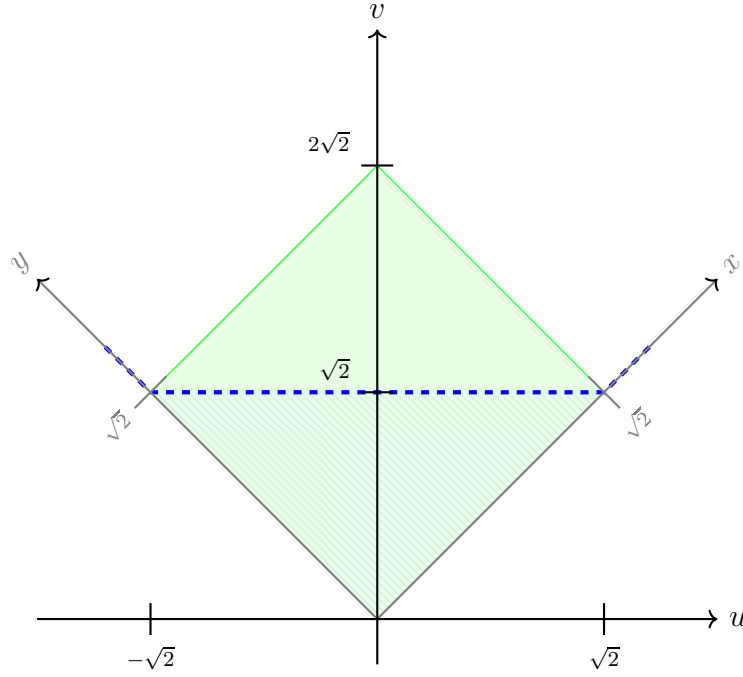


FIGURE 4. The green square  $S$  depicts the anisotropic Young diagram  $T(i^{\alpha i})$  in the Russian coordinate system. The blue hatched triangle depicts the limit shape  $\Lambda_\infty$ ; the blue dashed line depicts the corresponding profile  $\omega_{\Lambda_\infty}$ .

$\lambda_n$ . We will use the same transformation

$$T := T_{\sqrt{\frac{\alpha}{n}}, \sqrt{\frac{1}{\alpha n}}}$$

in order to scale both the original rectangular Young diagram  $(i^{\alpha i})$  as well as the resulting random Young diagram  $\lambda_n$ .

We notice that the anisotropic Young diagram  $T(i^{\alpha i})$  is a square, see Figure 4; we shall denote it by  $S$ . As we shall see in [DŚ17], the distribution of the random Young diagram  $\lambda_n$  can be equivalently formulated in terms of the corresponding natural reducible Jack character and Theorem 1.5 is applicable. Thus the sequence of anisotropic random Young diagrams  $\Lambda_n = T\lambda_n$  converges to some deterministic limit  $\Lambda_\infty$ . Not very surprisingly, this limit  $\Lambda_\infty$  turns out to be the bottom half of the square  $S$ , see Figure 4.

Figures 5 and 6 are an illustration of the law of large numbers (Theorem 1.6): as the number of boxes  $n \rightarrow \infty$  tends to infinity, suitably scaled random Young diagrams  $\Lambda_n$  indeed seem to converge to the deterministic limit  $\Lambda_\infty$ .

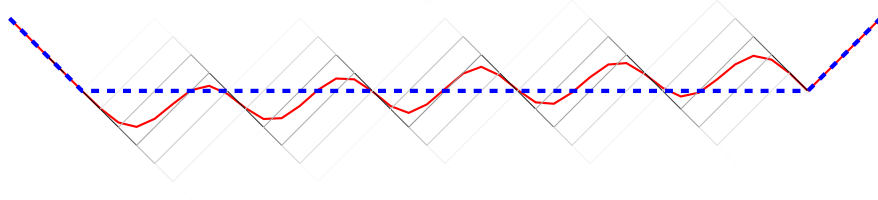


FIGURE 5. Diagonal gray lines form a *heatmap* showing probabilities that a given segment belongs to the profile of the random Young diagram  $\Lambda_n$  (the intensity of color corresponds to the probability). The rectangular grid of anisotropically stretched boxes is clearly visible. The blue dashed line depicts the limit profile  $\omega_{\Lambda_\infty}$ . In order to save space, only the neighborhood of  $\Lambda_\infty$  is shown. The red solid line depicts the mean value  $t \mapsto \mathbb{E}\omega_{\Lambda_n}(t)$ . In this example  $\alpha = 4$ ,  $i = 5$ ,  $n = 50$ .

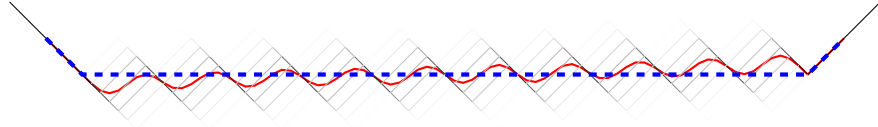


FIGURE 6. The analogue of Figure 5 for  $\alpha = 4$ ,  $i = 10$ ,  $n = 200$ . The increased number of boxes is compensated by a decrease of the size of the individual boxes. With this choice of scaling, the fluctuations of random Young diagrams  $\Lambda_n$  around  $\Lambda_\infty$  tend to zero as  $n \rightarrow \infty$ .

In order to speak about CLT one should consider a more refined scaling: the one in which one stretches the second Russian coordinate  $v$  by a factor of  $\sqrt{n}$ . This scaling has a bizarre feature: each individual box is drawn as a parallelogram in which the difference  $v_{\max} - v_{\min} = \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}$  of the  $v$ -coordinates of the top and the bottom vertex does not depend on  $n$  so one cannot claim that the *size* of an individual box converges to zero; nevertheless the *area* of an individual box does converge to zero. Figures 7–9 illustrate this choice of the scaling.

Theorem 1.7 states that the fluctuations of  $\Lambda_n$  *around its mean value* (which depends on  $n$ ) are asymptotically Gaussian; this calls for a more detailed investigation of the mean value  $\mathbb{E}\omega_{\Lambda_n}$ . This mean value has been indicated on Figures 5–9 by the red solid line. The first-order approximation

$$(1.20) \quad \lim_{n \rightarrow \infty} \mathbb{E}\omega_{\Lambda_n} = \omega_{\Lambda_\infty}$$

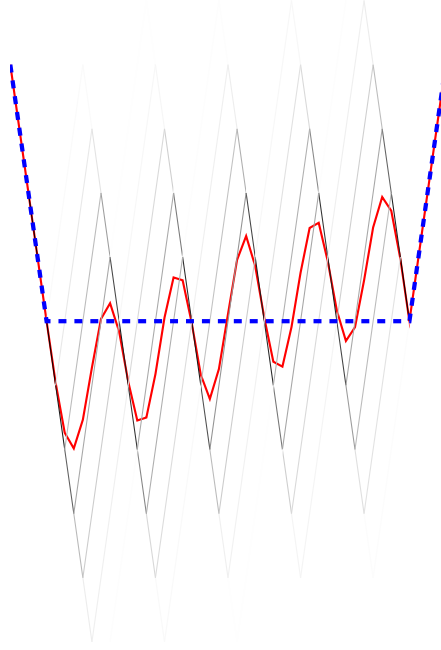


FIGURE 7. The analogue of Figure 5 for the profiles  $\sqrt{n} \omega_{\Lambda_n}$  for which only the second Russian coordinate  $v$  was anisotropically stretched by factor  $\sqrt{n}$ . In this example  $\alpha = 4, i = 5, n = 50$ .

is provided by Law of Large Numbers in Theorem 1.6 (to be very precise: the equality (1.20) does not follow from Theorem 1.6 itself but it does follow from its *proof*, as we shall see later). However, the scaling used in CLT and Figures 7–9 indicate that one should also understand the second-order asymptotics of the sequence of deterministic functions on the real line

$$(1.21) \quad \mathbb{E}\Delta_n := \sqrt{n} (\mathbb{E}\omega_{\Lambda_n} - \omega_{\Lambda_\infty}).$$

In fact, Figures 7–9 might indicate that it would be more natural to consider CLT in a formulation in which one would rather study the fluctuations of  $\Lambda_n$  around the limit shape  $\Lambda_\infty$  (which does not depend on  $n$ ) as opposed to the fluctuations around the mean value (which depends on  $n$ ). The former fluctuations are quantified by the random functions

$$(1.22) \quad \Delta_n := \sqrt{n} (\omega_{\Lambda_n} - \omega_{\Lambda_\infty}).$$

Unfortunately, in order to have the convergence of the sequence of mean values  $\mathbb{E}\Delta_n$  and convergence in distribution of the random functions (1.22) we will need more refined hypothesis than the one in Theorem 1.5. We postpone the discussion of this new hypothesis to Section 1.9 and we continue presentation of the CLT in this refined setting.

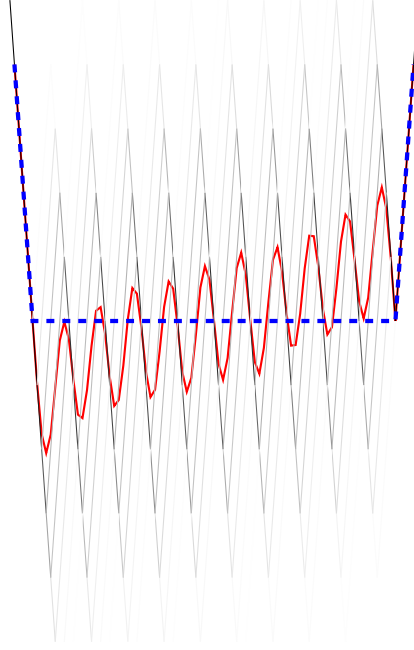


FIGURE 8. The analogue of Figure 7 for  $\alpha = 4$ ,  $i = 10$ ,  $n = 200$ . In this scaling the fluctuations of  $\sqrt{n} \omega_{\Lambda_n}$  (“the shaded area”) around  $\sqrt{n} \omega_{\Lambda_\infty}$  do not vanish as  $n \rightarrow \infty$ . Also the discrepancy between the mean value of these fluctuations  $\sqrt{n} \mathbb{E} \omega_{\Lambda_n}$  (the red solid line) and the ‘first-order approximation’  $\sqrt{n} \omega_{\Lambda_\infty}$  (the dashed blue line) does not vanish as  $n \rightarrow \infty$ .

#### 1.5.7. Central Limit Theorem, the refined version.

**Theorem 1.9** (Central Limit Theorem, the refined version). *Let  $(\chi_n)$  and  $(\lambda_n)$  be as in Theorem 1.6; we assume that  $(\chi_n)$  fulfills some additional hypothesis (‘enhanced approximate factorization property’, see Definition 1.11 later on).*

*Then the random vector  $\Delta_n$  converges in distribution to some (non-centered) Gaussian random vector  $\Delta_\infty$  as  $n \rightarrow \infty$ .*

Informally speaking, the above theorem states that asymptotically, for  $n \rightarrow \infty$

$$\omega_{\Lambda_n} \approx \omega_{\Lambda_\infty} + \frac{1}{\sqrt{n}} \Delta_\infty$$

where  $\omega_{\Lambda_\infty}$  is a deterministic curve and  $\Delta_\infty$  is a (non-centered) Gaussian process.

Just like Theorem 1.7, the true meaning of the above result can be only formulated with the help of some suitable test functions. In particular, we



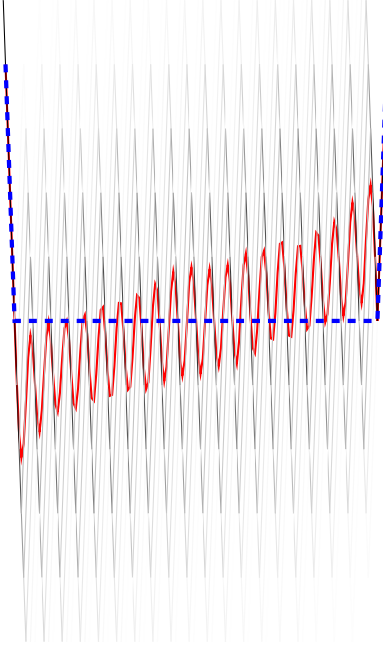


FIGURE 9. The analogue of Figure 7 for  $\alpha = 4$ ,  $i = 20$ ,  $n = 800$ . As  $n \rightarrow \infty$ , the amplitude of the oscillations of the discrepancy  $\sqrt{n} (\mathbb{E}\omega_{\Lambda_n} - \omega_{\Lambda_\infty})$  remain roughly constant, but their frequency tends to infinity and thus the discrepancy converges in the sense of Schwartz distributions.

will prove Theorem 1.9 by showing that the joint distribution of the family of random variables  $(Y_k)_{k \geq 2}$  converges as  $n \rightarrow \infty$  to a (non-centered) Gaussian distribution, where

$$(1.23) \quad Y_k := \frac{k-1}{2} \int u^{k-2} \Delta_n(u) du = \sqrt{n} \left( \mathcal{S}_k^{(1)}(\Lambda_n) - \mathcal{S}_k^{(1)}(\Lambda_\infty) \right)$$

is (up to a simple scalar factor) the value of the Schwartz distribution  $\Delta_n$  evaluated on a suitable polynomial test function.

The convergence in distribution in Theorem 1.9 (or, equivalently, the convergence in distribution of the family of random variables  $(Y_k)$ ) holds true both in the weak topology of probability measures as well as in moments. In particular, the mean value of the limit distribution

$$(1.24) \quad \mathbb{E}\Delta_\infty = \lim_{n \rightarrow \infty} \mathbb{E}\Delta_n$$

exists as a Schwartz distribution on the real line; the convergence holds in the weak sense, i.e. the limit

$$\int u^k \mathbb{E}\Delta_\infty(u) du := \lim_{n \rightarrow \infty} \int u^k \mathbb{E}\Delta_n(u) du$$

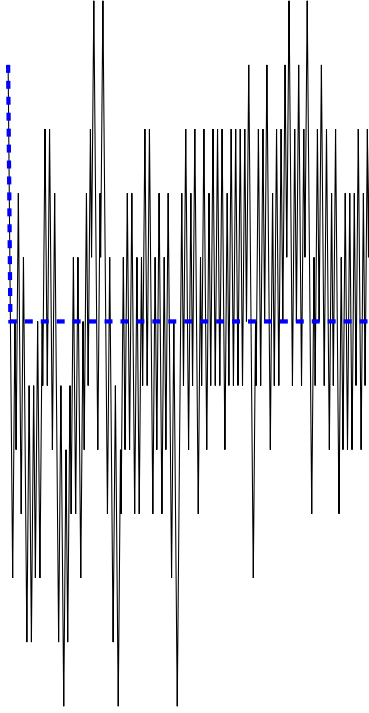


FIGURE 10. A sample profile  $\sqrt{n} \omega_{\Lambda_n}$  for  $\alpha = 4$ ,  $i = 80$ ,  $n = 12800$ .

exists and is finite for an arbitrary integer  $k \geq 0$ .

The proof of Theorem 1.9 is postponed to Section 3.10.

The convergence in (1.24) is illustrated in Figures 7–9: the function  $\mathbb{E}\Delta_n$  is the difference between the red solid and the blue dashed curves. As one can see on these examples, the function  $\mathbb{E}\Delta_n$  has oscillations of period and amplitude related to the grid of the boxes of the Young diagrams. As  $n \rightarrow \infty$ , the amplitude of these oscillations does not converge to zero (so that the convergence in the supremum norm does not hold) but their frequency tends to infinity (which is sufficient for convergence in the weak topology).

The central limit theorem which occurs in Theorems 1.7 and 1.9 is somewhat reminiscent to CLT for random walks. A significant difference lies in the nature of the limit object: in the case of the random walks it is the Brownian motion which has continuous trajectories while in the case considered in Theorems 1.7 and 1.9 it is a random Schwartz distribution  $\Delta_\infty$  which has quite singular ‘trajectories’, as it can be seen on a simulation shown in Figure 10.

**1.6. Preliminaries: partitions and permutations.** In order to discuss the details of the hypothesis of Theorem 1.5 we need to present some relationships between partitions and permutations. We use this occasion to introduce the notation related to them.

A *partition*  $\lambda = (\lambda_1, \dots, \lambda_l)$  is defined as a weakly decreasing finite sequence of positive integers. If  $\lambda_1 + \dots + \lambda_l = n$  we also say that  $\lambda$  is a *partition of  $n$*  and denote it by  $\lambda \vdash n$ . We also define

$$|\lambda| := \lambda_1 + \dots + \lambda_l.$$

For an integer  $n \geq 0$  we denote by  $\mathcal{P}_n$  the set of all partitions of the number  $n$ . We denote by

$$\mathcal{P} := \bigsqcup_{n \geq 0} \mathcal{P}_n$$

the set of all partitions.

Each partition can be represented graphically as an *Young diagram*. We denote the set of all Young diagrams with  $n$  boxes (i.e. the set of Young diagrams representing partitions of size  $n$ ) by  $\mathbb{Y}_n$  and we denote by  $\mathbb{Y}$  the set of all Young diagrams.

The expression *Young diagram* is fully synonymous to the expression *partition*. However, for aesthetical reasons, we will use each of them in a different context. For example, *partitions* will be used in order to enumerate conjugacy classes of the symmetric groups while *Young diagrams* in order to enumerate irreducible representations of the symmetric groups.

The empty partition as well as the empty Young diagram will be denoted by the same symbol  $\emptyset$ .

There is a natural bijective correspondence between the conjugacy classes in the symmetric group  $\mathfrak{S}(n)$  and the partitions of  $n$ . In particular, the identity permutation  $\text{id} \in \mathfrak{S}(n)$  corresponds to the partition

$$1^n = (1, \dots, 1) \in \mathcal{P}_n$$

which consists of  $n$  parts, each equal to 1. This correspondence between partitions and permutations will serve us as a powerful source of heuristics when dealing with partitions.

For a permutation  $\pi \in \mathfrak{S}(n)$  we denote by

$$\|\pi\| := n - \#\pi$$

its *length*, defined as the minimal number of factors necessary to write  $\pi$  as a product of transpositions. Above,  $\#\pi$  denotes the number of the cycles of the permutation  $\pi$ .

Analogously, for a partition  $\pi = (\pi_1, \dots, \pi_l) \in \mathcal{P}_n$  we define its *length*

$$\|\pi\| := |\pi| - \ell(\pi)$$

as the length of any permutation from the corresponding conjugacy class. Above  $\ell(\pi) := l$  denotes the *number of parts* of the partition  $\pi$ .

If  $k \leq n$  are positive integers, we can view any permutation  $\pi \in \mathfrak{S}(k)$  of the set  $[k] := \{1, 2, \dots, k\}$  also as a permutation  $\tilde{\pi} \in \mathfrak{S}(n)$  of the larger set  $[n] \supseteq [k]$  by declaring that  $\tilde{\pi}$  acts on the elements of the difference  $\{k+1, k+2, \dots, n\} = [n] \setminus [k]$  as the identity; in other words  $\tilde{\pi}$  is obtained from  $\pi$  by adding an appropriate number of fixpoints, thus we have a natural chain  $\mathfrak{S}(1) \subset \mathfrak{S}(2) \subset \dots$  of inclusions. If we pass to partitions, it follows that

$$(1.25) \quad \tilde{\pi} = (\pi, 1^{n-k}) \in \mathcal{P}_n$$

is the partition  $\pi$  augmented by an appropriate number of parts, each equal to 1. It is worth pointing out that partitions  $\tilde{\pi}$  and  $\pi$  have the same length:

$$\|\tilde{\pi}\| = \|\pi\|.$$

For partitions  $\pi_1, \pi_2, \dots, \pi_k$  we define their *product*  $\pi_1 \cdots \pi_k$  as their concatenation. This definition is motivated as follows: we can find some family  $A_1, \dots, A_k$  of disjoint sets such that  $A_i$  consists of  $|A_i| = |\pi_i|$  elements. We can also find a permutation  $\pi_i$  of the set  $A_i$  such that its cycle-decomposition is given by the partition  $\pi_i$ . By adding appropriate number of fixpoints, the domain of each permutation  $\pi_i$  can be extended to the union  $\bigsqcup_{1 \leq j \leq k} A_j$ . Then the cycle decomposition of the product  $\pi_1 \cdots \pi_k$  of permutations is indeed given by the product (i.e., concatenation)  $\pi_1 \cdots \pi_k$  of the partitions.

For a partition  $\lambda = (\lambda_1, \dots, \lambda_l)$  we denote by

$$m_i(\lambda) := |\{k : \lambda_k = i\}|$$

the *multiplicity* of  $i \geq 1$  in the partition  $\lambda$ . The numerical factor  $z_\lambda$  is defined by

$$(1.26) \quad z_\lambda := \prod_{i \geq 1} m_i(\lambda)! i^{m_i(\lambda)}.$$

**1.7. Approximate factorization of characters.** It is time to discuss the hypothesis of the main result, Theorem 1.5 and its concrete incarnations, Theorems 1.6, 1.7 and 1.9. This section is based on the ideas of Biane [Bia01] and the second-named author [Śni06b] who studied the first of the contexts from Section 1.1, i.e. random Young diagrams related to representations of the symmetric groups  $\mathfrak{S}(n)$ .

1.7.1. *Extension of the characters.* In the context of Theorem 1.5, for a given  $n \geq 1$  we start with a function  $\chi_n: \mathcal{P}_n \rightarrow \mathbb{R}$ . We extend the domain of  $\chi_n$  to the set  $\bigsqcup_{0 \leq k \leq n} \mathcal{P}_k$  of partitions of sufficiently small numbers by setting

$$(1.27) \quad \chi_n(\pi) := \chi_n(\pi, 1^{n-|\pi|}) \quad \text{for } |\pi| \leq n,$$

i.e. we extend the partition  $\pi$  by adding an appropriate number of parts, each equal to 1. For a heuristical motivation of this procedure we refer to Eq. (1.25) and the surrounding explanation of the relationship to permutations and the sequence of group inclusions  $\mathfrak{S}(1) \subset \mathfrak{S}(2) \subset \dots$ .

1.7.2. *The right decay of characters.* The first hidden assumption which appears in Theorem 1.5 is that the characters  $\chi_n$  converge to zero *with the right speed*. To be more specific: for any fixed partition  $\pi$  we require that

$$(1.28) \quad \chi_n(\pi) = O\left(n^{-\frac{\|\pi\|}{2}}\right)$$

in the limit as  $n \rightarrow \infty$ .

Note that thanks to (1.27) the left-hand side of (1.28) is well-defined for all sufficiently big values of  $n$ .

1.7.3. *Approximate factorization of characters.* The second hidden assumption which appears in Theorem 1.5 is that *the characters should approximately factorize*, i.e.

$$(1.29) \quad \chi_n(\pi_1 \cdots \pi_\ell) \approx \chi_n(\pi_1) \cdots \chi_n(\pi_\ell)$$

should hold true for any partitions  $\pi_1, \dots, \pi_\ell$ . The approximate equality (1.29) was formulated in a rather vague way; in the following we discuss it in more detail.

1.7.4. *Approximate factorization for two factors.* In the first non-trivial case  $\ell = 2$  the approximate equality (1.29) takes the following explicit form: for any partitions  $\pi_1$  and  $\pi_2$  we require that

$$(1.30) \quad \chi_n(\pi_1 \pi_2) - \chi_n(\pi_1) \chi_n(\pi_2) = O\left(n^{-\frac{\|\pi_1\| + \|\pi_2\|}{2} - 1}\right).$$

Note that our previous assumption (1.28) guarantees only that each of the two summands on the left-hand side is of order  $O\left(n^{-\frac{\|\pi_1\| + \|\pi_2\|}{2}}\right)$ , while the right-hand side of (1.30) gives a much better bound on their difference.

For a fixed value of  $n$ , it is convenient to regard the left-hand side of (1.30) as a kind of *covariance* in which the partitions  $\pi_1$  and  $\pi_2$  play the role of the random variables and the character  $\chi_n$  plays the role of the expected value:

$$\kappa_2(\pi_1, \pi_2) := \chi_n(\pi_1 \pi_2) - \chi_n(\pi_1) \chi_n(\pi_2).$$

This viewpoint will soon turn out to be very fruitful. In the following we shall explain how to generalize the notion of covariance to a bigger number of partitions.

**1.7.5. Cumulants in the classical probability theory.** In the classical probability theory one defines the *cumulant*  $\kappa^\mathbb{E}(y_1, \dots, y_\ell) = \kappa_\ell^\mathbb{E}(y_1, \dots, y_\ell)$  of random variables  $y_1, \dots, y_\ell$  as the solution of the following system of equations:

$$(1.31) \quad \left\{ \begin{array}{l} \mathbb{E}(x_1) = \kappa_1^\mathbb{E}(x_1), \\ \mathbb{E}(x_1 x_2) = \kappa_2^\mathbb{E}(x_1, x_2) + \kappa_1^\mathbb{E}(x_1) \kappa_1^\mathbb{E}(x_2), \\ \mathbb{E}(x_1 x_2 x_3) = \kappa_3^\mathbb{E}(x_1, x_2, x_3) + \kappa_1^\mathbb{E}(x_1) \kappa_2^\mathbb{E}(x_2, x_3) \\ \quad + \kappa_1^\mathbb{E}(x_2) \kappa_2^\mathbb{E}(x_1, x_3) + \kappa_1^\mathbb{E}(x_3) \kappa_2^\mathbb{E}(x_1, x_2) \\ \quad + \kappa_1^\mathbb{E}(x_1) \kappa_1^\mathbb{E}(x_2) \kappa_1^\mathbb{E}(x_3), \\ \vdots \end{array} \right.$$

which should hold for all choices of the random variables  $x_1, x_2, \dots$  which have all moments finite. In this system of equations, the symbol  $\mathbb{E}$  denotes the expected value; thus the left-hand side of the  $\ell$ -th equation is just the mixed moment of the random variables  $x_1, \dots, x_\ell$ . The summands on the right-hand side correspond to *set-partitions* (i.e., equivalence relations) on the set  $[\ell] := \{1, 2, \dots, \ell\}$ . Each factor corresponds to one of the blocks of the corresponding set-partition (i.e., to some equivalence class of the corresponding equivalence relation).

The system of equations (1.31) can be solved recursively, in particular

$$\kappa_1^\mathbb{E}(x_1) = \mathbb{E}(x_1)$$

is just the mean value, while

$$\kappa_2^\mathbb{E}(x_1, x_2) = \mathbb{E}(x_1 x_2) - \mathbb{E}(x_1) \mathbb{E}(x_2)$$

is the covariance.

Alternatively, the cumulant might be defined as an appropriate coefficient of the formal expansion of the logarithm of the multidimensional Laplace transform:

$$(1.32) \quad \kappa_\ell^\mathbb{E}(x_1, \dots, x_\ell) = \frac{\partial^\ell}{\partial x_1 \dots \partial x_\ell} \log \mathbb{E} e^{t_1 x_1 + \dots + t_\ell x_\ell} \Big|_{t_1 = \dots = t_\ell = 0}.$$

Informally speaking, higher order cumulants  $\kappa_\ell^\mathbb{E}(x_1, \dots, x_\ell)$  for  $\ell \geq 2$  quantify *to which extent random variables  $x_1, \dots, x_\ell$  fail to be independent*

or, in other words, *to which extent the mixed moment  $\mathbb{E}(x_1 \cdots x_\ell)$  does not factor*.

**1.7.6. Cumulants of partitions.** We will adapt the notion of cumulants to the framework considered in the current article. The role of the sequence  $x_1, x_2, \dots$  of random variables will be played by a sequence  $\pi_1, \pi_2, \dots$  of partitions. Also, the role of the expected value  $\mathbb{E}$  will be played by some function  $\chi: \mathcal{P} \rightarrow \mathbb{R}$  on the set of partitions. Thus for the purposes of this introduction the *cumulants* are defined as the solution of the following system of equations:

$$(1.33) \quad \left\{ \begin{array}{l} \chi(\pi_1) = \kappa_1^\chi(\pi_1), \\ \chi(\pi_1 \pi_2) = \kappa_2^\chi(\pi_1, \pi_2) + \kappa_1^\chi(\pi_1) \kappa_1^\chi(\pi_2), \\ \chi(\pi_1 \pi_2 \pi_3) = \kappa_3^\chi(\pi_1, \pi_2, \pi_3) + \kappa_1^\chi(\pi_1) \kappa_2^\chi(\pi_2, \pi_3) \\ \quad + \kappa_1^\chi(\pi_2) \kappa_2^\chi(\pi_1, \pi_3) + \kappa_1^\chi(\pi_3) \kappa_2^\chi(\pi_1, \pi_2) \\ \quad + \kappa_1^\chi(\pi_1) \kappa_1^\chi(\pi_2) \kappa_1^\chi(\pi_3), \\ \vdots \end{array} \right.$$

We will pay special attention to the case when each partition  $\pi_i = (l_i)$  consists of exactly one part  $l_i \geq 2$ . In this case the system (1.33) takes the following, even more concrete form:

$$\left\{ \begin{array}{l} \chi(l_1) = \kappa_1^\chi(l_1), \\ \chi(l_1, l_2) = \kappa_2^\chi(l_1, l_2) + \kappa_1^\chi(l_1) \kappa_1^\chi(l_2), \\ \chi(l_1, l_2, l_3) = \kappa_3^\chi(l_1, l_2, l_3) + \kappa_1^\chi(l_1) \kappa_2^\chi(l_2, l_3) \\ \quad + \kappa_1^\chi(l_2) \kappa_2^\chi(l_1, l_3) + \kappa_1^\chi(l_3) \kappa_2^\chi(l_1, l_2) \\ \quad + \kappa_1^\chi(l_1) \kappa_1^\chi(l_2) \kappa_1^\chi(l_3), \\ \vdots \end{array} \right.$$

On the left-hand side we use the simplified notation that

$$\chi(l_1, \dots, l_\ell) := \chi((l_1, \dots, l_\ell))$$

denotes the evaluation  $\chi(\pi)$  on the partition  $\pi := (l_1, \dots, l_\ell)$ . Analogously, on the right-hand side we use the simplified notation

$$\kappa_\ell^\chi(l_1, \dots, l_\ell) := \kappa_\ell^\chi((l_1), \dots, (l_\ell)) \quad \text{for } l_1, \dots, l_\ell \geq 2.$$

**1.8. Hypothesis of CLT.** The rather informal requirement (1.29) which was the hidden hypothesis of Theorem 1.7 can now be made more explicit as follows.

*Definition 1.10.* Assume that for each integer  $n \geq 1$  we are given a function  $\chi_n: \mathcal{P}_n \rightarrow \mathbb{R}$ . We say that the sequence  $\chi_1, \chi_2, \dots$  has *approximate factorization property* if for each integer  $\ell \geq 1$  and all integers  $l_1, \dots, l_\ell \geq 2$  the limit

$$(1.34) \quad \lim_{n \rightarrow \infty} \kappa_\ell^{\chi_n}(l_1, \dots, l_\ell) n^{\frac{l_1 + \dots + l_\ell + \ell - 2}{2}} = \lim_{n \rightarrow \infty} \kappa_\ell^{\chi_n}(l_1, \dots, l_\ell) n^{\frac{\|(l_1)\| + \dots + \|(l_\ell)\| + 2(\ell - 1)}{2}}$$

exists and is finite.

**1.9. Hypothesis of CLT, refined version.** We are now ready to state the hidden technical assumption behind the more refined version of CLT (Theorem 1.9).

*Definition 1.11.* Assume that for each integer  $n \geq 1$  we are given a function  $\chi_n: \mathcal{P}_n \rightarrow \mathbb{R}$ . We say that the sequence  $\chi_1, \chi_2, \dots$  has *enhanced approximate factorization property* if it fulfills approximate factorization property from Definition 1.10 and, additionally, in the case  $\ell = 1$  the rate of convergence in (1.34) takes the following explicit form: for each  $l \geq 2$  there exist some constants  $a_{l+1}, b_{l+1} \in \mathbb{R}$  such that

$$(1.35) \quad \kappa_1^{\chi_n}(l) n^{\frac{l-1}{2}} = a_{l+1} + \frac{b_{l+1} + o(1)}{\sqrt{n}} \quad \text{for } n \rightarrow \infty.$$

**1.10. Example: Jack–Plancherel measure.** The power of our main result (Theorem 1.5 and, more precisely, Definiton-Theorem 2.4) lies in the fact that there is a lot of natural examples which fulfill the hypothesis.

*Example 1.12.* We continue Example 1.4, where Jack–Plancherel measure was considered. For each value of  $n$  the character  $\chi_n: \mathcal{P}_n \rightarrow \mathbb{R}$  given by (1.4) has the property that for all integers  $\ell \geq 1$  and  $l_1, \dots, l_\ell \geq 2$

$$\chi_n(l_1, \dots, l_\ell) = 0;$$

it follows immediately that all cumulants

$$\kappa_\ell^{\chi_n}(l_1, \dots, l_\ell) = 0$$

tautologically vanish, except for  $\ell = 1$  and  $l_1 = 1$ .

It follows that the sequence  $(\chi_n)$  has approximate factorization property (also in the enhanced version) and Theorems 1.7 and 1.9 apply (Theorem 1.6 applies as well, as we shall see after its hypothesis is presented in Section 2.8.2). This example has been studied by Dołęga and Féray [DF16] who proved the Gaussianity of fluctuations for this particular example.



For more examples see Section 1.13.

**1.11. What happens when  $\alpha \neq 1$ ?** In the formulation of Theorems 1.5, 1.6, 1.7 and 1.9 we tried to avoid the details of the relationship between

(a) the asymptotics of the characters  $(\chi_n)$

on one side, and

(b<sub>1</sub>) the limit shape  $\Lambda_\infty$  of the random Young diagrams  $(\Lambda_n)$ ,

(b<sub>2</sub>) the covariance of the Gaussian process  $\Delta_\infty$  which describes the fluctuations of  $\Lambda_n$  around the limit  $\Lambda_\infty$ , and

(b<sub>3</sub>) the Schwartz distribution  $\mathbb{E}\Delta_\infty$

on the other side. It is the time to address this issue and, in particular, *to compare this relationship with a better-studied case  $\alpha = 1$*  [Bia98, Bia01, Śni06b]. We summarize below the discussion from Section 3.

**1.11.1. Law of Large Numbers.** The relationship between the asymptotics of characters  $(\chi_n)$  and the limit shape  $\Lambda_\infty$  which appears in Theorem 1.6 *does not* depend on the value of  $\alpha$ ; in particular it coincides with the findings of Biane [Bia98, Bia01] for the special case  $\alpha = 1$ .

**1.11.2. Central Limit Theorem, the simpler version.** Also the limiting Gaussian distribution of the family of centered random variables  $(X_k)$  from Theorem 1.7 *does not* depend on the value of  $\alpha$ ; in particular its covariance coincides with the one found by the second-named author [Śni06b] for the special case  $\alpha = 1$ .

**1.11.3. Central Limit Theorem, the refined version.** On the other hand, the Schwartz distribution  $\mathbb{E}\Delta_\infty$  from Theorem 1.9 which provides more refined asymptotics of the discrepancy between the Young diagrams  $\Lambda_n$  and the limit shape  $\Lambda_\infty$  *depends* on the value of the deformation parameter  $\alpha$ . In particular, for  $\alpha = 1$  in a typical application when the asymptotics (1.35) holds true with  $b_l \equiv 0$  we have that  $\mathbb{E}\Delta_\infty = 0$  and the non-centeredness of the limit Gaussian distribution  $\Delta_\infty$  does not occur.

This new phenomenon of non-centeredness  $\mathbb{E}\Delta_\infty \neq 0$  has been observed by the first-named author and Féray [DF16] in the special case of Jack-deformed Plancherel measure from Examples 1.4 and 1.12, and its origin in the general case is discussed in Section 3.8.

**1.12. Double scaling limit.** Until now we have kept the deformation parameter  $\alpha$  constant as the number of boxes  $n \rightarrow \infty$  tends to infinity. *What can we say about the asymptotics of random Young diagrams if  $\alpha = \alpha(n)$  depends on the number of boxes?* A partial answer to this question is given by the following result.

**Theorem 1.13.** *Assume that  $\alpha = \alpha(n)$  is a function of the number of boxes of the Young diagrams with the property that the limit*

$$(1.36) \quad g := \lim_{n \rightarrow \infty} \frac{-\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}}{\sqrt{n}}$$

*exists and is finite. Then Theorems 1.6 and 1.7 remain valid.*

*Assume that more refined asymptotics*

$$(1.37) \quad \frac{-\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}}{\sqrt{n}} = g + \frac{g'}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

*holds true for  $n \rightarrow \infty$  for some constant  $g'$ . Then Theorem 1.9 remains valid.*

This theorem will not require a separate proof because Theorems 1.6, 1.7 and 1.9 will be proved from the very beginning in this more general setup.

Note that the case when  $\alpha$  is a constant corresponds to

$$g = 0, \quad g' = -\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}.$$

An interesting choice is  $g > 0$  and  $g' = 0$  with

$$(1.38) \quad \alpha(n) := \frac{1}{g^2 n}$$

because the width and the height of the box given by (1.14) correspond to

$$w = \frac{1}{gn}, \quad h = g,$$

i.e. the width of a box tends to zero as the number of the boxes tends to infinity while its height remains constant. We will concentrate on this scaling in the following.

The anisotropic Young diagram  $\Lambda_n$  from (1.15) obtained in this way (in the French convention) is a collection of rectangles of the same height  $g$  and of the widths

$$(1.39) \quad \frac{\lambda_1}{gn}, \frac{\lambda_2}{gn}, \dots;$$

see Figure 11 for an example. This opens up the possibility of describing the asymptotics of shapes of the Young diagrams in terms of the quantities (1.39) instead of the profile  $\omega_{\Lambda_n}$ . We leave it as an open problem.

It is not very surprising that with this kind of scaling, the lack of the dependence on the deformation parameter  $\alpha$  of the limit shape  $\Lambda_\infty$  (which we discussed in Section 1.11.1), as well as of the covariance of the fluctuations (which we discussed in Section 1.11.2) *does not* hold true anymore, see Section 3.

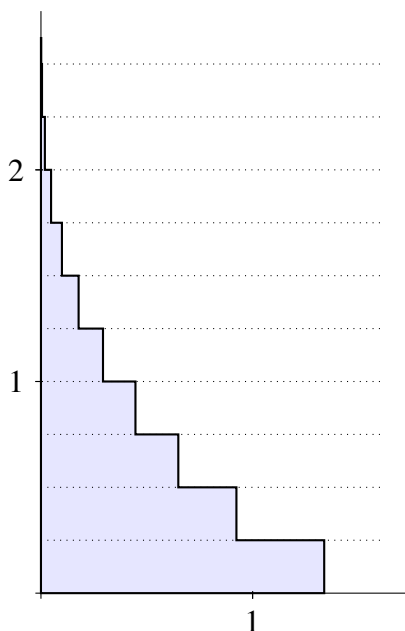


FIGURE 11. Limit shape of random Young diagrams distributed according to the Jack–Plancherel measure in the double scaling limit for  $g = \frac{1}{4}$ .

**1.13. More examples. The forthcoming paper [DŚ17].** The class of reducible Jack characters with the (enhanced) approximate factorization property for which Theorems 1.6, 1.7 and 1.9 are applicable contains many natural examples. For a simple concrete example see Section 1.10; we devote the whole forthcoming paper [DŚ17] to the investigation of this class and presentation of even more examples.

For instance, there is a concrete reducible Jack character with the (enhanced) approximate factorization property, which leads to a one-parameter deformation of the *Schur–Weyl measures* [Bia01, Śni06b, Mé11] which originate in the representation theory of the symmetric groups  $\mathfrak{S}(n)$  and the representation theory of the general linear groups  $\mathrm{GL}_d$ .

The second-named author [Śni06b] proved that in the special case  $\alpha = 1$  the class of reducible characters of the symmetric groups with the approximate factorization property is closed with respect to three natural operations which originate in the representation theory of the symmetric groups: *the restriction, the induction and the outer product*. In [DŚ17] we adapt these three operations to the setup of Jack characters and we prove that an analogue of the aforementioned result from [Śni06b] remain true in the new

setup. In other words: the class of reducible Jack characters with the (enhanced) approximate factorization property is closed with respect to three natural operations. This allows to produce new examples of random Young diagrams to which the results of the current paper are applicable from the old ones; in particular the example which we have studied in Section 1.5.6 fits into this category.

**1.14. Content of the paper.** In Section 2 we introduce the main algebraic tool for our considerations and we provide several equivalent characterizations of the approximate factorization property of characters in Theorem 2.4. We apply them to prove Theorem 1.7 and to prove the first part of Theorem 1.6. In Section 3 we prove our main tool, that is Theorem 2.4, and we prove Theorem 1.9. As a byproduct, we gather all the necessary tools to finish the proof of Theorem 1.6, which is presented in Section 4.

## 2. APPROXIMATE FACTORIZATION OF CHARACTERS

The purpose of this section is to give a number of conditions which are equivalent to the approximate factorization property (Definition 1.10). These conditions often turn out to be more convenient in applications, such as the ones from [DŚ17].

In order to study the shape of a (random) Young diagram  $\lambda$  we will need some convenient family  $(F_i)$  of functionals  $\lambda \mapsto F_i(\lambda)$ . Investigation of a random Young diagram  $\lambda$  corresponds to investigation of the joint distribution of the family of random variables  $(F_i(\lambda))$ . In Section 2.3 we shall introduce a convenient family  $\mathcal{P}$  of such functionals.

**2.1. Normalized Jack characters.** The usual way of viewing the characters of the symmetric groups is to fix the irreducible representation  $\lambda$  and to consider the character as a function of the conjugacy class  $\pi$ . However, there is also another very successful viewpoint due to Kerov and Olshanski [KO94], called *dual approach*, which suggests to do roughly the opposite. Lassalle [Las08, Las09] adapted this idea to the framework of Jack characters. In order for this dual approach to be successful (both with respect to the usual characters of the symmetric groups and for the Jack characters) one has to choose the most convenient normalization constants. In the current paper we will use the normalization introduced by Dołęga and Féray [DF16] which offers some advantages over the original normalization of Lassalle. Thus, with the right choice of the multiplicative constant, the unnormalized Jack character  $\chi_\lambda^{(\alpha)}$  from (1.9) becomes the *normalized Jack character*  $\text{Ch}_\pi^{(\alpha)}(\lambda)$ , defined as follows.

*Definition 2.1.* Let  $\alpha > 0$  be given and let  $\pi$  be a partition. For any Young diagram  $\lambda$  the value of the *normalized Jack character*  $\text{Ch}_\pi^{(\alpha)}(\lambda)$  is given by:

$$(2.1) \quad \text{Ch}_\pi^{(\alpha)}(\lambda) := \begin{cases} |\lambda|^{\frac{|\pi|}{\alpha}} \chi_\lambda^{(\alpha)}(\pi) & \text{if } |\lambda| \geq |\pi|; \\ 0 & \text{if } |\lambda| < |\pi|, \end{cases}$$

where

$$n^{\underline{k}} := n(n-1) \cdots (n-k+1)$$

denotes the falling power and  $\chi_\lambda^{(\alpha)}(\pi)$  is the Jack character (1.9). The choice of an empty partition  $\pi = \emptyset$  is acceptable; in this case  $\text{Ch}_\emptyset^{(\alpha)}(\lambda) = 1$ .

Each Jack character depends on the deformation parameter  $\alpha$ ; in order to keep the notation light we will make this dependence implicit and we will simply write  $\text{Ch}_\pi(\lambda)$ .

*Remark 2.2.* We continue the discussion from Remark 1.3 about the choice of the multiplicative constants in the definition of the irreducible Jack characters  $\chi_\pi$ . In the case of more established quantities, the normalized Jack characters  $\text{Ch}_\pi$ , the choice of the normalization constants has been a subject of careful discussion [DF16]. In fact, thanks to this normalization, Jack characters  $\text{Ch}_\pi$  can be defined and characterized in an abstract way which does not refer to the notion of Jack polynomials, see the result of Féray [Śni15, Theorem A.2]. With this in mind we can reverse the logic and we can view the equality (2.1), which gives the relationship between the normalized characters  $\text{Ch}_\pi = \text{Ch}_\pi^{(\alpha)}$  and their non-normalized counterparts  $\chi_\pi^{(\alpha)}$ , as a *definition* of the irreducible Jack character  $\chi_\pi^{(\alpha)}$ . This does not answer the original question about the motivations for the particular choice of the normalization constants of  $\chi_\pi^{(\alpha)}$  but transforms it into the question about the motivations standing behind the relationship (2.1). The latter equality stems from the case  $\alpha = 1$  and we decided to keep it in the same exact form without any additional factors depending on the deformation parameter  $\alpha$ , because such factors would affect the formulation of the key result Theorem 2.4, particularly in the double scaling limit when  $\alpha = \alpha(n)$  tends to zero or infinity. This is the *a priori* motivation for the bizarre choice of the normalization constants in (1.9) which we promised in Remark 1.3.

**2.2. The deformation parameters.** In order to avoid dealing with the square root of the variable  $\alpha$  which is ubiquitous in the subject of Jack deformation, we introduce an indeterminate  $A := \sqrt{\alpha}$ . The algebra of Laurent polynomials in the indeterminate  $A$  will be denoted by  $\mathbb{Q}[A, A^{-1}]$ .

A special role will be played by the quantity

$$(2.2) \quad \gamma := -A + \frac{1}{A} \in \mathbb{Q}[A, A^{-1}]$$

which already appeared in the double scaling limit (1.36).

**2.3. The algebra  $\mathcal{P}$  of polynomial functions.** In recent papers [Śni15, Śni16] the second-named author has defined a certain filtered algebra  $\mathcal{P}$ , which is called *the algebra of polynomial functions*. This algebra consists of certain functions in the set  $\mathbb{Y}$  of Young diagrams with values in the ring  $\mathbb{Q}[A, A^{-1}]$  of Laurent polynomials and, among many equivalent definitions, one can define it using normalized Jack characters.

*Definition 2.3* ([Śni15, Proposition 2.17] and [Śni16, Proposition 1.7]). The algebra  $\mathcal{P}$  is defined as a linear span (with rational coefficients) of the functions

$$(2.3) \quad \gamma^k \text{Ch}_\pi : \mathbb{Y} \rightarrow \mathbb{Q}[A, A^{-1}]$$

over the integers  $k \geq 0$  and over partitions  $\pi \in \mathcal{P}$ , where  $\gamma$  was defined in (2.2). The multiplication in  $\mathcal{P}$  is defined as pointwise multiplication of the functions on  $\mathbb{Y}$ . The filtration on  $\mathcal{P}$  is specified as follows: for an integer  $n \geq 0$  the set of elements of degree at most  $n$  is defined as the linear span of the elements (2.3) over the integers  $k \geq 0$  and over partitions  $\pi \in \mathcal{P}$  such that

$$k + |\pi| + \ell(\pi) \leq n.$$

The heuristic explanation of this particular choice of the filtration on  $\mathcal{P}$  comes from an observation that — as we shall see later — for sequences  $(\chi_n)$  of characters which we consider in the current paper, the corresponding mean value (which will be defined quite soon, in (2.5))

$$\mathbb{E}_{\chi_n}(x) = O\left(n^{\frac{\deg x}{2}}\right) \quad \text{for } n \rightarrow \infty$$

tends to infinity at the speed specified by the degree of a given  $x \in \mathcal{P}$ , also in the more general scaling (1.36).

**2.4. Algebra  $\mathcal{P}_\bullet$  of polynomial functions with the disjoint product  $\bullet$ .** The set  $\mathcal{P}$  of polynomial functions can be equipped with another multiplication  $\bullet$ , called *disjoint product* [Śni16, Section 1.9], which is defined on the linear base of Jack characters by *concatenation* (see the end of Section 1.6) of the corresponding partitions

$$(\gamma^p \text{Ch}_\pi) \bullet (\gamma^q \text{Ch}_\sigma) := \gamma^{p+q} \text{Ch}_{\pi\sigma}.$$

It is easy to check that this product is commutative and associative; the set of polynomial functions equipped with this multiplication becomes an algebra which will be denoted by  $\mathcal{P}_\bullet$ .

It is easy to check that the usual filtration on the algebra  $\mathcal{P}$  works fine with this product; in this way  $\mathcal{P}_\bullet$  becomes a filtered algebra.

### 2.5. Probabilistic structures on the algebra of polynomial functions.

Assume that  $\chi: \mathcal{P}_n \rightarrow \mathbb{R}$  is a reducible Jack character and let  $\mathbb{P}_\chi$  be the corresponding probability measure (1.10) on the set  $\mathbb{Y}_n$  of Young diagrams with  $n$  boxes. With this setup functions on  $\Omega := \mathbb{Y}_n$  can be viewed as random variables; we denote by  $\mathbb{E}_\chi$  the corresponding expectation.

Let us fix a partition  $\pi \vdash n$ ; we denote by  $\chi_\#^{(\alpha)}(\pi)$  the random variable  $\mathbb{Y}_n \ni \lambda \mapsto \chi_\lambda^{(\alpha)}(\pi)$  given by irreducible Jack character (1.9). From the way the probability measure  $\mathbb{P}_\chi$  was defined in (1.10) it follows immediately that

$$(2.4) \quad \mathbb{E}_\chi \left[ \chi_\#^{(\alpha)}(\pi) \right] = \chi(\pi).$$

Any polynomial function  $F \in \mathcal{P}$  can be restricted to the set  $\mathbb{Y}_n$  of Young diagrams with  $n$  boxes; thus it makes sense to speak about its expected value  $\mathbb{E}_\chi F$ . In the case when  $F = \text{Ch}_\pi$  is a Jack character, this expected value can be explicitly calculated thanks to (2.4):

$$(2.5) \quad \mathbb{E}_\chi \text{Ch}_\pi = \begin{cases} |\lambda|^{\frac{|\pi|}{|\lambda|}} \chi(\pi) & \text{if } |\lambda| < |\pi|, \\ 0 & \text{otherwise.} \end{cases}$$

The expected value  $\mathbb{E}_\chi$  gives rise via (1.31) to the corresponding cumulants  $\kappa_\ell^\chi$ .

Recall that the set of polynomial functions can be equipped with an alternative multiplicative structure given by the disjoint product  $\bullet$ . The expected value does not depend on the choice of the multiplicative structure hence the corresponding expected value  $\mathbb{E}_\chi: \mathcal{P}_\bullet \rightarrow \mathbb{R}$  coincides with the expected value  $\mathbb{E}_\chi: \mathcal{P} \rightarrow \mathbb{R}$  considered above. However, the choice of the multiplicative structure has an impact on the cumulants; the corresponding cumulants in  $\mathcal{P}_\bullet$  will be denoted by  $\kappa_{\bullet\ell}^\chi$ .

### 2.6. Equivalent characterizations of approximate factorization of characters.

The following result, Theorem 2.4, is the key tool for the purposes of the current paper. Its main content is part (a); roughly speaking, it states that each of the four families of numbers (2.6)–(2.9) can be transformed into the others. Each of these four families describes some convenient aspect of the characters  $\chi_n$  in the limit  $n \rightarrow \infty$ . To be more specific:

- The family (2.8) (and its subset, the family (2.7)) has a direct probabilistic meaning. It contains information about the cumulants of some random variables which might be handy while proving probabilistic statements such as Central Limit Theorem or Law of Large Numbers.

- On the other hand, the cumulants appearing in the families (2.6) and (2.9) are purely algebraic and do *not* have any direct probabilistic meaning. However, their merit lies in the fact that in many concrete applications (such as the ones from [DŚ17]) it is much simpler to verify algebraic conditions (A) and (D) than their probabilistic counterparts (B) and (C).

**Theorem 2.4** (The key tool). *Assume that  $\alpha > 0$  is fixed or, more generally, that  $\alpha = \alpha(n)$  is such as in the double scaling regime from Theorem 1.13. Assume also that for each integer  $n \geq 1$  we are given a reducible Jack character  $\chi_n: \mathcal{P}_n \rightarrow \mathbb{R}$ .*

- (a) Equivalent characterization of approximate factorization property (Definition 1.10). *Then the following four conditions are equivalent:*

(A) *for each integer  $\ell \geq 1$  and all integers  $l_1, \dots, l_\ell \geq 2$  the limit*

$$(2.6) \quad \lim_{n \rightarrow \infty} \kappa_\ell^{\chi_n}((l_1), \dots, (l_\ell)) n^{\frac{l_1 + \dots + l_\ell + \ell - 2}{2}} =$$

$$\lim_{n \rightarrow \infty} \kappa_\ell^{\chi_n}((l_1), \dots, (l_\ell)) n^{\frac{\|(l_1)\| + \dots + \|(l_\ell)\| + 2(\ell - 1)}{2}}$$

*exists and is finite (for the definition of such cumulants see Section 1.7.6);*

(B) *for each integer  $\ell \geq 1$  and all  $x_1, \dots, x_\ell \in \{\text{Ch}_2, \text{Ch}_3, \dots\}$  the limit*

$$(2.7) \quad \lim_{n \rightarrow \infty} \kappa_\ell^{\chi_n}(x_1, \dots, x_\ell) n^{-\frac{\deg x_1 + \dots + \deg x_\ell - 2(\ell - 1)}{2}}$$

*exists and is finite (for the definition of such cumulants see Section 2.5);*

(C) *for each integer  $\ell \geq 1$  and all  $x_1, \dots, x_\ell \in \mathcal{P}$  the limit*

$$(2.8) \quad \lim_{n \rightarrow \infty} \kappa_\ell^{\chi_n}(x_1, \dots, x_\ell) n^{-\frac{\deg x_1 + \dots + \deg x_\ell - 2(\ell - 1)}{2}}$$

*exists and is finite;*

(D) *for each integer  $\ell \geq 1$  and all  $x_1, \dots, x_\ell \in \mathcal{P}_\bullet$  the limit*

$$(2.9) \quad \lim_{n \rightarrow \infty} \kappa_{\bullet\ell}^{\chi_n}(x_1, \dots, x_\ell) n^{-\frac{\deg x_1 + \dots + \deg x_\ell - 2(\ell - 1)}{2}}$$

*exists and is finite.*



- (b) Equivalent characterization of enhanced approximate factorization property (Definition 1.11). *Assume that the conditions from part (a) hold true. Furthermore, assume that for  $\ell = 1$  the rate of the convergence of any of the four expressions under the limit symbol in (2.6)–(2.9) is of the form*

$$(2.10) \quad \text{const}_1 + \frac{\text{const}_2 + o(1)}{\sqrt{n}}$$

*in the limit  $n \rightarrow \infty$  and all choices of  $l_1$  (respectively, for all choices of  $x_1$ ); the constants depend on the choice of  $l_1$  (respectively,  $x_1$ ).*

*Then for  $\ell = 1$  the rate of convergence of each of the four expressions (2.6)–(2.9) is of the form (2.10).*

When  $\alpha = 1$ , part (a) of the above result corresponds to [Śni06b, Theorem and Definition 1]. The proof is postponed to Section 3.

In concrete applications it is convenient to know the relationships between the asymptotics of each the four quantities (2.6)–(2.9); we postpone the details of these relationships to Section 3.

**2.7. Application of the key tool: Gaussianity of fluctuations. Proof of Theorem 1.7.** We start by revisiting Section 1.5 where the *conclusion* part of Theorem 1.5 was discussed.

**2.7.1. Functionals of shape, revisited.** For a Young diagram  $\lambda$ , a real number  $\alpha > 0$  and an integer  $k \geq 2$  we define

$$\mathcal{S}_k^{(\alpha)}(\lambda) := \mathcal{S}_k^{(1)} \left( T_{\sqrt{\alpha}, \frac{1}{\sqrt{\alpha}}} \lambda \right) = (k-1) \iint_{(x,y) \in \lambda} \left( \sqrt{\alpha} x - \frac{1}{\sqrt{\alpha}} y \right)^{k-2} dx dy,$$

where  $\mathcal{S}_k^{(1)}$  has been defined in (1.17) and the integral on the right-hand side is taken over a polygon on the plane defined by the Young diagram  $\lambda$  (drawn in the French convention). In order to keep the notation light we shall make the dependence on  $\alpha$  implicit and we shall simply write  $\mathcal{S}_k(\lambda) := \mathcal{S}_k^{(\alpha)}(\lambda)$ .

One can show [Śni15, Proposition 2.9] that  $\mathcal{S}_k \in \mathcal{P}$  is a polynomial function of degree  $k$ .

A simple linear change of the variables in the integrals implies that for  $\Lambda_n$  given by (1.15) we have

$$(2.11) \quad \mathcal{S}_k^{(1)}(\Lambda_n) = \frac{1}{n^{\frac{k}{2}}} \mathcal{S}_k(\lambda_n).$$

As a consequence, the random variable  $X_k$  from (1.19) can be written as

$$X_k = \frac{1}{n^{\frac{k-1}{2}}} (\mathcal{S}_k(\lambda_n) - \mathbb{E}\mathcal{S}_k(\lambda_n)),$$

where  $\lambda_n$  is a random Young diagram from Theorem 1.7. If we take the viewpoint from Section 2.5 and we view  $\mathcal{S}_k \in \mathcal{P}$  as a random variable then we can write even simpler

$$(2.12) \quad X_k = \frac{1}{n^{\frac{k-1}{2}}} (\mathcal{S}_k - \mathbb{E}\mathcal{S}_k).$$

**2.7.2. Proof of Theorem 1.7.** From the very beginning we shall consider the more general setup of the double scaling limit from Theorem 1.13 in which the deformation parameter  $\alpha = \alpha(n)$  might depend on  $n$ .

The informal Central Limit Theorem from Theorem 1.5 was formalized in Theorem 1.7 (see also Remark 1.8) as a statement about the limiting probability distribution of the family of random variables  $(X_i)$  from (1.19) and (2.12) in the limit  $n \rightarrow \infty$ . In order to prove this result we shall investigate the cumulants of the form

$$\kappa_\ell(X_{i_1}, \dots, X_{i_\ell}).$$

We first notice that for  $\ell = 1$  such a cumulant

$$\kappa_1(X_{i_1}) = \mathbb{E}X_{i_1} = 0$$

is equal to zero by definition of centered random variables.

Consider now the case  $\ell \geq 2$ . In this case the cumulant  $\kappa_\ell$  is translation-invariant; from (2.12) it follows therefore that

$$(2.13) \quad \kappa_\ell(X_{i_1}, \dots, X_{i_\ell}) = \kappa_\ell\left(\frac{1}{n^{\frac{i_1-1}{2}}} \mathcal{S}_{i_1}, \dots, \frac{1}{n^{\frac{i_\ell-1}{2}}} \mathcal{S}_{i_\ell}\right).$$

The hypothesis of Theorem 1.7 is that  $(\chi_n)$  has approximate factorization property (Definition 1.10). By Theorem 2.4(a) the latter is equivalent to condition (C) which we apply in the special case when  $(x_1, \dots, x_\ell) := (\mathcal{S}_{i_1}, \dots, \mathcal{S}_{i_\ell})$ . The latter implies that the right-hand side of (2.13) is of order  $O\left(n^{\frac{2-\ell}{2}}\right)$ . This implies that for each  $\ell \geq 3$

$$\lim_{n \rightarrow \infty} \kappa_\ell(X_{i_1}, \dots, X_{i_\ell}) = 0.$$

Consider now the case  $\ell = 2$ . If we adapt the above reasoning, we get that the limit

$$\lim_{n \rightarrow \infty} \kappa_2(X_{i_1}, X_{i_2})$$

exists and is finite.

Let us summarize the above discussion. We have proved that the limit

$$\lim_{n \rightarrow \infty} \kappa_\ell (X_{i_1}, \dots, X_{i_\ell})$$

exists and is finite for any choice of  $\ell \geq 1$  and  $i_1, \dots, i_\ell \geq 2$ . In other words: the joint distribution of the random variables  $(X_i)$  converges in moments as  $n \rightarrow \infty$  to the joint distribution of an abstract family of random variables  $(Z_i)$  with the property that all cumulants vanish:  $\kappa_\ell (Z_{i_1}, \dots, Z_{i_\ell}) = 0$ , except for  $\ell = 2$ . The latter is the defining property of the Gaussian centered distribution. Since the Gaussian distribution is uniquely determined by its moments, it follows that  $(X_i)$  converges to  $(Z_i)$  not only in moments but also in the weak topology of probability measures, as required.  $\square$

## 2.8. Application of the key tool: Law of large numbers. Proof of Theorem 1.6.

**2.8.1. Measure associated with a Young diagram.** Suppose an anisotropic Young diagram  $\Lambda \subseteq \mathbb{R} \times \mathbb{R}$  (viewed in the French coordinate system) is given. We will assume for simplicity that the area of  $\Lambda$  is equal to 1. Let  $(x, y) \in \Lambda$  be a random point in  $\Lambda$ , sampled with the uniform probability. We denote by  $P_\Lambda$  the probability distribution of its Russian coordinate

$$u = x - y.$$

It is a probability measure on  $\mathbb{R}$  with the probability density

$$(2.14) \quad f_\Lambda(u) = \frac{\omega_\Lambda(u) - |u|}{2}.$$

This density is a Lipschitz function with the Lipschitz constant equal to 1. Such probability measures  $P_\Lambda$  will be our main tool for investigation of asymptotics of Young diagrams  $\Lambda$ . The Reader should be advised that this is *not* Kerov's transition measure which is also a probability measure on the real line associated with a Young diagram [Ker93b] for similar purposes.

Any anisotropic Young diagram  $\Lambda$  with unit area which contains some point  $(x_0, y_0)$ , contains also the whole rectangle  $\{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0\}$ ; by comparison of the areas it follows that  $x_0 y_0 \leq 1$ . The latter inequality written in the Russian coordinate system gives the following restriction on the possible values of the corresponding profile  $\omega$ :

$$|u| \leq \omega(u) \leq \sqrt{u^2 + 4}$$

and for the corresponding density

$$(2.15) \quad 0 \leq f_\Lambda(u) \leq \frac{\sqrt{u^2 + 4} - |u|}{2} = \frac{2}{\sqrt{u^2 + 4} + |u|}.$$

By (1.17) the moments of the measure  $P_\Lambda$  are related to the values of the functionals of shape, as follows:

$$\int u^k dP_\Lambda(u) = \frac{1}{k+1} \mathcal{S}_{k+2}^{(1)}(\Lambda).$$

**2.8.2. Details of the hypothesis and proof of Theorem 1.6.** We are now ready to state the missing details of the hypothesis of Theorem 1.6.

**Hypothesis 2.5.** Assume that we are given a sequence  $(\chi_n)$  of reducible Jack characters  $\chi_n: \mathcal{P}_n \rightarrow \mathbb{R}$  which fulfills approximate factorization property (Definition 1.10) with  $\alpha = \alpha(n)$  as in (1.36). We assume additionally that

$$\sup_{l \geq 2} \frac{\sqrt[l]{|a_l|}}{l^m} < \infty,$$

where

$$a_{l+1} := \lim_{n \rightarrow \infty} \chi_n(l) n^{\frac{l-1}{2}} \quad \text{for } l \geq 1$$

gives the first-order asymptotics of the characters evaluated on cycles, and

$$(2.16) \quad m = \begin{cases} 2 & \text{for } g \neq 0, \\ 1 & \text{for } g = 0, \end{cases}$$

where  $g$  is given by (1.36).

Note that in the generic case  $g \neq 0$  the assumptions are much weaker than in the specific case  $g = 0$ .

We can now proceed with the proof of Theorem 1.6.

*The first part of the proof of Theorem 1.6.* From the very beginning we shall consider the more general setup of the double scaling limit from Theorem 1.13 in which the deformation parameter  $\alpha = \alpha(n)$  might depend on  $n$ .

**Random variables  $S_k^{[n]}$  and their convergence in probability.** We continue the discussion from Section 2.7.2. For  $k \geq 2$  consider the random variable

$$S_k^{[n]} := \mathcal{S}_k^{(1)}(\Lambda_n) = \frac{1}{n^{\frac{k}{2}}} \mathcal{S}_k(\lambda_n).$$

By Hypothesis 2.5, the condition (A) from Theorem 2.4 is fulfilled; it follows that the condition (C) is fulfilled as well. In the special case  $\ell = 1$  and  $x_1 = \mathcal{S}_k$  it follows that the limit

$$(2.17) \quad s_k := \lim_{n \rightarrow \infty} \mathbb{E}_{\chi_n} S_k^{[n]} = \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{k}{2}}} \mathbb{E}_{\chi_n} \mathcal{S}_k$$

exists; in the special case  $\ell = 2$  and  $x_1 = x_2 = \mathcal{S}_k$  it follows that the variance

$$\text{Var } S_k^{[n]} = \kappa_2^{\chi_n} \left( \frac{1}{n^{\frac{k}{2}}} \mathcal{S}_k, \frac{1}{n^{\frac{k}{2}}} \mathcal{S}_k \right) = O \left( \frac{1}{n} \right)$$

converges to zero. Chebyshev's inequality implies that for each  $k \geq 2$  the sequence of random variables  $(S_k^{[n]})_n$  converges (as  $n \rightarrow \infty$ ) to  $s_k$  in probability.

**The limiting probability measure  $P_{\Lambda_\infty}$ .** By (2.15) the mean value

$$(2.18) \quad u \mapsto \mathbb{E} f_{\Lambda_n}(u)$$

exists, is finite, and fulfills analogous bounds to (2.15). It is the density of the probability measure  $\mathbb{E} P_{\Lambda_n}$ ; the moments of this measure are given by

$$\int u^k d\mathbb{E} P_{\Lambda_n}(u) = \frac{1}{k+1} \mathbb{E}_{\chi_n} S_{k+2}^{[n]}.$$

The topology on the set of probability measures with all moments finite given by convergence of moments can be metrized; we denote by  $d$  the corresponding distance. Equation (2.17) implies that the sequence of measures  $(\mathbb{E} P_{\Lambda_n})$  is a Cauchy sequence in the metric space given by  $d$ ; the sequence  $(\mathbb{E} P_{\Lambda_n})$  converges therefore *in moments* to some probability measure which will be denoted by  $P_{\Lambda_\infty}$ , in particular

$$\int u^k dP_{\Lambda_\infty}(u) = \frac{1}{k+1} s_{k+2}.$$

In general, it might happen that the measure  $P_{\Lambda_\infty}$  is not unique; it turns out, however, that in the setup which we consider *the measure  $P_{\Lambda_\infty}$  is uniquely determined by its moments*. We postpone the proof of this result until Section 4.5 once we gather the necessary tools in Section 3. In the remaining part of the proof we will make extensive use of this (unproved yet) result. Since  $P_{\Lambda_\infty}$  is determined by its moments, it follows that  $(\mathbb{E} P_{\Lambda_n})$  converges to  $P_{\Lambda_\infty}$  *in the weak topology of probability measures*.

**Weak convergence of probability measures implies uniform convergence of densities.** For  $\epsilon > 0$  and  $u_0 \in \mathbb{R}$  let  $\phi_\epsilon: \mathbb{R} \rightarrow \mathbb{R}_+$  be a function on the real line such that  $\phi_\epsilon$  is supported on an  $\epsilon$ -neighborhood of  $u_0$  and

$\int \phi_\epsilon(u) du = 1$ . Since  $\mathbb{E}f_{\Lambda_n}$  is Lipschitz with constant 1, it follows that

$$(2.19) \quad \left| \mathbb{E}f_{\Lambda_n}(u_0) - \int \phi_\epsilon(u) d\mathbb{E}P_{\Lambda_n}(u) \right| = \left| \int \phi_\epsilon(u) (\mathbb{E}f_{\Lambda_n}(u_0) - \mathbb{E}f_{\Lambda_n}(u)) du \right| \leq \int \phi_\epsilon(u) |\mathbb{E}f_{\Lambda_n}(u_0) - \mathbb{E}f_{\Lambda_n}(u)| du \leq \epsilon.$$

By weak convergence of probability measures, the integral on the left-hand side converges, as  $n \rightarrow \infty$ , to  $\int \phi_\epsilon(u) dP_{\Lambda_\infty}(u)$ . By passing to the limit, the above inequality implies therefore that

$$\limsup_{n \rightarrow \infty} \left| \mathbb{E}f_{\Lambda_n}(u_0) - \int \phi_\epsilon(u) dP_{\Lambda_\infty}(u) \right| \leq \epsilon.$$

Since this inequality holds true for arbitrary  $\epsilon > 0$ , it follows that the sequence  $\mathbb{E}f_{\Lambda_n}(u_0)$  is a Cauchy sequence, hence it converges to a finite limit which will be denoted by  $f_{\Lambda_\infty}(u_0)$ . In other words, we have proved that the functions  $\mathbb{E}f_{\Lambda_n}$  converge pointwise to the function  $f_{\Lambda_\infty}$ ; since all functions  $\mathbb{E}f_{\Lambda_n}$  are Lipschitz with the same constant, *the convergence is uniform on a compact set  $K = [-R, R]$  for arbitrary value of  $R$ .*

On the other hand, inequalities (2.15) show that the distance between  $\mathbb{E}f_{\Lambda_n}$  and  $f_{\Lambda_\infty}$  with respect to the supremum norm on the set  $K^c = \mathbb{R} \setminus K$  is bounded by

$$(2.20) \quad \|\mathbb{E}f_{\Lambda_n} - f_{\Lambda_\infty}\|_{L^\infty[K^c]} \leq \frac{2}{\sqrt{R^2 + 4} + |R|}.$$

Since the right-hand side converges to zero as  $R \rightarrow \infty$ , it follows that *the sequence of functions  $\mathbb{E}f_{\Lambda_n}$  converges to  $f_{\Lambda_\infty}$  uniformly on the whole real line  $\mathbb{R}$ .*

**Convergence of densities, in probability.** We have proved that the sequence of random variables  $(S_k^{[n]})_n$  converges (as  $n \rightarrow \infty$ ) to  $s_k$  in probability; in other words for each  $\epsilon > 0$  and each integer  $k \geq 0$

$$(2.21) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left( \left| \int u^k dP_{\Lambda_n}(u) - \int u^k dP_{\Lambda_\infty}(u) \right| > \epsilon \right) = 0;$$

and the sequence of random probability measures  $P_{\Lambda_n}$  converges to the measure  $P_{\Lambda_\infty}$  *in moments, in probability.*

The topology of convergence in the weak topology of probability measures can be metrized by Lévy–Prokhorov distance  $\pi$ . Since the measure

$P_{\Lambda_\infty}$  is uniquely determined by its moments, convergence to  $P_{\Lambda_\infty}$  *in moments* implies convergence to the same limit *in the weak topology of probability measures*. With the help of the distances  $d$  and  $\pi$  the latter statement can be rephrased as follows: for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any probability measure  $\mu$

$$d(\mu, P_{\Lambda_\infty}) < \delta \implies \pi(\mu, P_{\Lambda_\infty}) < \varepsilon.$$

It follows that the sequence of random probability measures  $P_{\Lambda_n}$  converges to the measure  $P_{\Lambda_\infty}$  *in the weak topology of probability measures, in probability*, i.e. for each  $\varepsilon > 0$

$$(2.22) \quad \lim_{n \rightarrow \infty} \mathbb{P}(\pi(P_{\Lambda_n}, P_{\Lambda_\infty}) > \varepsilon) = 0.$$

Let  $\epsilon > 0$ ; by adapting the proof of (2.19), we get that

$$(2.23) \quad \left| f_{\Lambda_n}(u_0) - \int \phi_\epsilon(u) dP_{\Lambda_n}(u) \right| \leq \epsilon.$$

Lévy–Prokhorov distance  $\pi$  metrizes the weak convergence of probability measures; it follows that we can choose sufficiently small  $\varepsilon > 0$  with the property that for any probability measure  $\mu$  on  $\mathbb{R}$

$$(2.24) \quad \pi(\mu, P_{\Lambda_\infty}) \leq \varepsilon \implies \left| \int \phi_\epsilon(u) d\mu(u) - \int \phi_\epsilon(u) dP_{\Lambda_\infty}(u) \right| < \epsilon.$$

Equation (2.22) combined with (2.24) as well as (2.23) imply therefore that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \left| f_{\Lambda_n}(u_0) - \int \phi_\epsilon(u) dP_{\Lambda_\infty}(u) \right| > 2\epsilon \right) = 0.$$

As  $\epsilon \searrow 0$ , the integral  $\int \phi_\epsilon(u) dP_{\Lambda_\infty}(u)$  converges to the density  $f_{\Lambda_\infty}(u_0)$  by an analogue of (2.19). In this way we proved that *for each  $u_0$  the sequence  $f_{\Lambda_n}(u_0)$  converges to  $f_{\Lambda_\infty}(u_0)$  in probability*.

By the same type of argument as in (2.20) it follows that *the sequence of functions  $f_{\Lambda_n}$  converges uniformly to  $f_{\Lambda_\infty}$  in probability*, i.e. for each  $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}(\|f_{\Lambda_n} - f_{\Lambda_\infty}\|_\infty > \epsilon) = 0$$

with respect to the supremum norm.

**Back to Young diagrams.** The function  $\omega_{\Lambda_\infty}$  which was promised in the formulation of Theorem 1.6 is simply given by the relationship (2.14) for the specific choice of  $f_\Lambda := f_{\Lambda_\infty}$ , namely

$$\omega_{\Lambda_\infty}(u) := 2f_{\Lambda_\infty}(u) + |u|.$$

Keeping in mind that we still have to prove that the measure  $P_{\Lambda_\infty}$  is uniquely determined by its moments (this proof is postponed until Section 4.5 once we gather the necessary tools in Section 3) we just finished

the proof of the fact that  $\omega_{\Lambda_n}$  converges to  $\omega_{\Lambda_\infty}$  in the supremum norm as  $n \rightarrow \infty$ , in probability, thus the proof of Theorem 1.6 is completed.  $\square$

### 3. FINE DETAILS: RELATIONSHIPS BETWEEN VARIOUS ALGEBRAIC AND PROBABILISTIC QUANTITIES

The content of this section is twofold.

Firstly, we shall present the proof of the key tool, Theorem 2.4. This result is very similar to a previous result of the second-named author [Śni06b, Theorem and Definition 1] which concerns the special case  $\alpha = 1$ ; for this reason we will present only the details how to adapt this older proof into our more general setup.

Secondly, we shall also discuss the missing details from the statement of Theorem 2.4 which also happen to be the missing details of the main result of the current paper, Theorem 1.5 and its concrete incarnations (Theorems 1.6, 1.7 and 1.9). To be more specific:

- concerning part (a) of Theorem 2.4 we shall discuss the exact relationship between the limits of the quantities (2.6)–(2.9) in the case  $\ell \in \{1, 2\}$ . This relationship also provides the information about the limit shape of random Young diagrams in Theorem 1.6 as well as about the covariance of the limit Gaussian process describing the fluctuations in Theorem 1.7.
- concerning part (b) of Theorem 2.4 we shall discuss the exact relationship between the constants which describe the fine asymptotics (2.10) of the quantities (2.6)–(2.9) in the case  $\ell = 1$ . This relationship also provides the information about the more refined asymptotics encoded by the Schwartz distribution  $\mathbb{E}\Delta_\infty$  from Theorem 1.9.

**3.1. Conditional cumulants.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be commutative unital algebras and let  $\mathbb{E}: \mathcal{A} \rightarrow \mathcal{B}$  be a unital linear map. We will say that  $\mathbb{E}$  is a *conditional expectation value*; in the literature one usually imposes some additional constraints on the structure of  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathbb{E}$ , but for the purposes of the current paper such additional assumptions will not be necessary.

In concordance with (1.32), for any tuple  $x_1, \dots, x_\ell \in \mathcal{A}$  we define their *conditional cumulant* as

$$\kappa_{\mathcal{A}}^{\mathcal{B}}(x_1, \dots, x_\ell) = [t_1 \cdots t_\ell] \log \mathbb{E} e^{t_1 x_1 + \cdots + t_\ell x_\ell} = \left. \frac{\partial^\ell}{\partial t_1 \cdots \partial t_\ell} \log \mathbb{E} e^{t_1 x_1 + \cdots + t_\ell x_\ell} \right|_{t_1 = \cdots = t_\ell = 0} \in \mathcal{B},$$

where the operations on the right-hand side should be understood in the sense of formal power series in variables  $t_1, \dots, t_\ell$ .



### 3.2. Approximate factorization property for polynomial functions.

*Definition 3.1.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be filtered, commutative, unital algebras and let  $\mathbb{E} : \mathcal{A} \rightarrow \mathcal{B}$  be a conditional expectation. We say that  $\mathbb{E}$  has *approximate factorization property* if for all choices of  $x_1, \dots, x_\ell \in \mathcal{A}$  we have that

$$\deg \kappa_{\mathcal{A}}^{\mathcal{B}}(x_1, \dots, x_\ell) \leq \deg x_1 + \dots + \deg x_\ell - 2(\ell - 1).$$

We consider the filtered unital algebras  $\mathcal{P}_\bullet$  and  $\mathcal{P}$  from Sections 2.3 and 2.4, and as a conditional expectation between them we take the identity map:

$$(3.1) \quad \mathcal{P}_\bullet \xrightarrow{\text{id}} \mathcal{P}.$$

This structure may appear to be misleadingly simple. Note, however, that even though  $\mathcal{P}$  and  $\mathcal{P}_\bullet$  are isomorphic as *vector spaces*, they are furnished with different multiplication structures. For this reason this identity map is far from being trivial.

We denote by  $\kappa_\bullet$  the conditional cumulants related to the conditional expectation (3.1). Informally speaking, this cumulant quantifies how *big* is the difference between the two multiplications on the set of polynomial functions: the disjoint product  $\bullet$  and the usual pointwise multiplication of functions.

We are now ready to state the main auxiliary result, proved very recently by the second-named author, which will be necessary for the proof of the key tool, Theorem 2.4.

**Theorem 3.2** ([Śni16, Theorem 1.9]). *The identity map*

$$(3.2) \quad \mathcal{P}_\bullet \xrightarrow{\text{id}} \mathcal{P}.$$

*has approximate factorization property.*

**3.3. Approximate factorization for  $\alpha = 1$ .** We denote by  $\mathcal{P}^{(1)}$  a version of the filtered algebra of polynomial functions  $\mathcal{P}$  obtained by the specialization  $\alpha := 1$ ; analogously we denote by  $\mathcal{P}_\bullet^{(1)}$  the algebra  $\mathcal{P}^{(1)}$  equipped with the multiplication given by the disjoint product.

The following result has been proved by the second-named author.

**Theorem 3.3** ([Śni06b, Theorem 15]). *The identity map*

$$(3.3) \quad \mathcal{P}_\bullet^{(1)} \xleftarrow{\text{id}} \mathcal{P}^{(1)}.$$

*has approximate factorization property.*

Theorems 3.2 and 3.3 are of the same flavor. There are two major differences between them: firstly, the arrows in (3.2) and (3.3) point in the opposite directions; secondly, the algebra  $\mathcal{P}$  is more rich than its specialized version  $\mathcal{P}^{(1)}$ , in particular the variable  $\gamma \in \mathcal{P}$  is not treated like a scalar since  $\deg \gamma = 1 > 0$ . For a detailed discussion of the relationship between these two results we refer to [Śni16, Section 1.13.2].

**3.4. Proof of Theorem 2.4, part (a).** The original proof of an analogue of part (a) of Theorem 2.4 for  $\alpha = 1$  [Śni06b, Theorem and Definition 1] was based on an application of Theorem 3.3. However, if one replaces all invocations of this result by Theorem 3.2 and, occasionally, replaces the roles played by  $\mathcal{P}$  and  $\mathcal{P}_\bullet$ , one obtains a valid proof of Theorem 2.4(a).

Some additional care has to be taken while proving that the condition (A) implies the remaining three conditions from part (a) of Theorem 2.4 which is related to the issue that the variable  $\gamma = \gamma(n)$  might depend on  $n$ , the number of boxes of the considered Young diagrams. However, in the double scaling limit which we consider (cf. Theorem 1.13) we still have that  $\gamma = O\left(n^{\frac{\deg \gamma}{2}}\right)$  which does not create any difficulties.

*Remark 3.4.* We use this occasion in order to state an erratum: the statement of [Śni06b, Corollary 19] is *not* correct. Roughly speaking, this result states that if a condition of the form (B) from Theorem 2.4 holds true for all  $x_1, \dots, x_\ell \in X$ , where  $X$  is a set which generates the algebra  $\mathcal{P}$ , then the same condition holds for *arbitrary*  $x_1, \dots, x_\ell$ . This assumption about  $X$  is not sufficient; the right assumption is that  $X$  generates  $\mathcal{P}$  in a way that is compatible with the filtration: we should require that each element  $z \in \mathcal{P}$  can be expressed as a linear combination of monomials in  $X$  with the property that the degree of *each* monomial is bounded from above by the degree of  $z$ . However, this erratum does not invalidate the proofs from [Śni06b] since all in applications of [Śni06b, Corollary 19] the above condition was fulfilled.

**3.5. Free cumulants.** In our presentation of the Central Limit Theorem we decided to parametrize the shape of the Young diagram  $\lambda$  by the functionals of shape  $\mathcal{S}_k^{(1)}$  from Section 1.5.4 or, preferentially, in terms of the functionals  $\mathcal{S}_k(\lambda)$  from Section 2.7.1. These functionals have an advantage of being conceptually very simple.

However, in many calculations related to the asymptotic representation theory it is more convenient to parametrize the shape of the Young diagram  $\lambda$  by *free cumulants*  $\mathcal{R}_k(\lambda)$ . In the context of the representation theory of the symmetric groups these quantities have been introduced by Biane [Bia98]. For the purposes of the current paper it is enough to know that

for a fixed Young diagram  $\lambda$  its sequence of functionals of shape and its sequence of free cumulants are related to each other by the following simple systems of equations [DFŚ10, Eqs. (14) and (15)]:

$$(3.4) \quad \mathcal{S}_l = \sum_{i \geq 1} \frac{1}{i!} (l-1)_{i-1} \sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = l}} \mathcal{R}_{k_1} \cdots \mathcal{R}_{k_i}, \quad l \geq 2,$$

$$(3.5) \quad \mathcal{R}_l = \sum_{i \geq 1} \frac{1}{i!} (-l+1)^{i-1} \sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = l}} \mathcal{S}_{k_1} \cdots \mathcal{S}_{k_i}, \quad l \geq 2,$$

where we use a shorthand notation  $\mathcal{S}_k = \mathcal{S}_k(\lambda)$ ,  $\mathcal{R}_k = \mathcal{R}_k(\lambda)$ . For a more in-depth discussion of the above relationships as well as a conceptual explanation of their analytic and combinatorial meaning we refer to [DFŚ10, Sections 2 and 3]. In fact, the above-cited papers [Bia98, DFŚ10] concerned only the special isotropic case  $\alpha = 1$ , however the passage to the anisotropic case does not create any difficulties, see the work of Lassalle [Las09] (who used a different normalization constants) as well as of the first-named author and Féray [DF16] (whose normalization we use).

**3.6. The case  $\alpha = 1$ .** The main advantage of free cumulants lies in the combination of the following two facts.

- Each free cumulant  $\mathcal{R}_k$  of a given Young diagram  $\lambda$  can be efficiently calculated [Bia98] and its dependence on the shape of  $\lambda$  takes a particularly simple form (more specifically,  $\mathcal{R}_k$  is a *homogeneous* function of degree  $k$ , see [Śni15, Lemma 2.10]).
- The family  $(\gamma, \mathcal{R}_2, \mathcal{R}_3, \dots)$  forms a convenient algebraic basis of the algebra  $\mathcal{P}$  with  $\deg \gamma = 1$  and  $\deg \mathcal{R}_k = k$ . In the special case  $\alpha = 1$  (which corresponds to  $\gamma = 0$ ) the expansion of  $\text{Ch}_l$  in this basis takes the following, particularly simple form:

$$(3.6) \quad \text{Ch}_l = \mathcal{R}_{l+1} + (\text{terms of degree at most } l-1).$$

One of the consequences of (3.6) is that in the special case  $\alpha = 1$  the relationship announced in Theorem 2.4(b) between the refined asymptotics of the four quantities (2.6)–(2.9) for  $\ell = 1$  takes the following, particularly simple form. Assume that there exists some sequence  $(a_l)$  with the property that

$$(3.7) \quad \chi_n(l) = a_{l+1} n^{-\frac{l-1}{2}} + O\left(n^{-\frac{l+1}{2}}\right) \quad \text{for all } l \geq 1;$$

note that it is a stronger version of (2.10) with  $\text{const}_2 \equiv 0$ ; then

$$(3.8) \quad \mathbb{E}_{\chi_n}(\text{Ch}_l) = a_{l+1} n^{\frac{l+1}{2}} + O\left(n^{\frac{l-1}{2}}\right) \quad \text{for all } l \geq 1;$$

$$(3.9) \quad \mathbb{E}_{\chi_n}(\mathcal{R}_{l+1}) = a'_{l+1} n^{\frac{l+1}{2}} + O\left(n^{\frac{l-1}{2}}\right) \quad \text{for all } l \geq 1,$$

$$(3.10) \quad \mathbb{E}_{\chi_n}(\mathcal{S}_l) = a''_l n^{\frac{l}{2}} + O\left(n^{\frac{l-2}{2}}\right) \quad \text{for each } l \geq 2,$$

where

$$(3.11) \quad a'_{l+1} = a_{l+1}$$

and

$$(3.12) \quad a''_l = \sum_{i \geq 1} \frac{1}{i!} (l-1)_{i-1} \sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = l}} a'_{k_1} \cdots a'_{k_i},$$

see (3.4) for the last equality and [Śni06b] for more details.

To summarize: for  $\alpha = 1$  it is easy to choose the initial sequence of the characters  $(\chi_n)$  in such a way that the subleading asymptotic terms in Eqs. (3.7)–(3.10) are relatively small, namely of order  $\frac{1}{n}$  times the leading asymptotic terms. As we shall see later in the proof of Theorem 1.9 in Section 3.10, this rapid decay of the subleading terms in (3.10) implies that the corresponding deterministic shift  $\mathbb{E}\Delta_\infty$  which describes fine-scale asymptotics of the mean shape of the random Young diagrams vanishes and  $\mathbb{E}\Delta_\infty = 0$ .

**3.7. Details of Theorem 2.4 part (a) in the generic case  $\alpha \neq 1$ .** The original proof of [Śni06b, Theorem and Definition 1] was based on the idea of expressing various elements  $F$  of the algebra of polynomial functions  $\mathcal{P}^{(1)}$  (such as the characters  $\text{Ch}_\pi$  or the conditional cumulants  $\kappa_\bullet$  of such characters) as polynomials in the basis  $\mathcal{R}_2, \mathcal{R}_3, \dots$  of free cumulants and studying the top-degree of such polynomials, see Section 3.6. In our context of  $\alpha \neq 1$  (or, in other words,  $\gamma \neq 0$ ) the corresponding polynomial for  $F$  might have some extra terms which depend additionally on the variable  $\gamma$ . These extra terms might influence the asymptotic behavior of random Young diagrams. We shall discuss this issue in more detail in the remaining of this section as well as in Section 3.9.

**3.7.1. The first-order asymptotics when  $\alpha$  is constant.** The first-named author and Féray [DF16, Proposition 3.7] proved that the degree of

$$F := \text{Ch}_l$$

with respect to the grading from Section 3.6 remains equal to  $l + 1$  even when we pass from  $\mathcal{P}^{(1)}$  to  $\mathcal{P}$  or, in other words, that the degree of the extra terms is bounded from above by  $l + 1$ . In the asymptotics when  $\alpha$  is a constant and does not depend on  $n$ , it follows that  $\gamma = O(1) \ll O\left(n^{\frac{\deg \gamma}{2}}\right)$ . Since each extra term is divisible by the monomial  $\gamma$ , it follows that the contribution of the extra terms is negligible when compared to the unique original top-degree term  $\mathcal{R}_{l+1}$ . It follows that in this asymptotics not only the proof of [Śni06b, Theorem and Definition 1] remains valid in our context, but *also the relationships between the quantities  $(a_l)$ ,  $(a'_l)$  and  $(a''_l)$  which provide the first-order asymptotics of, respectively, the characters, the mean value of free cumulant, and the mean value of the functionals of shape, remain the same as in Section 3.6.*

From Theorem 3.2 it follows that an analogous result holds true for

$$F := \kappa_{\bullet}(\text{Ch}_{l_1}, \text{Ch}_{l_2})$$

and the degree of  $F$  remains equal to  $l_1 + l_2$  when we pass from  $\mathcal{P}^{(1)}$  to  $\mathcal{P}$ . A reasoning similar to the one above implies that *the covariance of the Gaussian process describing the fluctuations of random Young diagrams coincides with its counterpart for  $\alpha = 1$  from [Śni06b, Theorem 3].*

**3.7.2. The first-order asymptotics in the double scaling limit.** In the asymptotics when  $\alpha = \alpha(n)$  depends on  $n$  in a way described in Theorem 1.13, the extra terms in both examples considered above are of the same order as the original terms. It follows that the proof of [Śni06b, Theorem and Definition 1] still remains valid in our context but the relationship between the quantities  $(a_l)$ ,  $(a'_l)$  and  $(a''_l)$  is altered and depends on the constant  $g$  from (1.36), see Section 3.9.2 below. Also the covariance of the Gaussian process describing the fluctuations of random Young diagrams is altered; finding an explicit form for this covariance is currently beyond our reach because no closed formula for the top-degree part of the conditional cumulant  $\kappa_{\bullet}(\text{Ch}_{l_1}, \text{Ch}_{l_2})$  (an analogue of the results of [Śni15] for  $\text{Ch}_n$ ) is available.

**3.8. Refined asymptotics of characters.** In order to perform the program outlined in Section 3.7 we need an analogue of the relationship (3.6) between the character  $\text{Ch}_l$  and the free cumulants in the case  $\alpha \neq 1$ . We present below two such formulas: the one from Section 3.8.1 is conceptually simpler and will be sufficient for the scaling when  $\alpha$  is fixed; for the purposes of the double scaling limit we will need a more involved formula from Section 3.8.2.

3.8.1. *The rough estimate.* We present here the formula expressing the top-degree part of the normalized Jack character  $\text{Ch}_l$  modulo terms divisible by  $\gamma^2$ . It turns out that this approximation of the top-degree part is sufficient for studying asymptotic behaviour of characters when  $\alpha$  is fixed.

$$(3.13) \quad \text{Ch}_l = \left[ \mathcal{R}_{l+1} + \sum_{i \geq 1} \sum_{k_1 + \dots + k_i = l} \frac{l}{i} (k_1 - 1) \cdots (k_i - 1) \mathcal{R}_{k_1} \cdots \mathcal{R}_{k_i} + \right. \\ \left. + (\text{terms divisible by } \gamma^2) \right] + (\text{terms of degree at most } l - 1).$$

The above formula follows from [Las09, Section 11] combined with the degree bounds of the first-named author and Féray [DF16, Proposition 3.7] as well as from [Śni15, Corollary 0.5].

3.8.2. *Closed formula for top-degree part of Jack characters.* Let us fix an integer  $l \geq 1$ . We will view the symmetric group  $\mathfrak{S}(l)$  as the set of permutations of the set  $[l] := \{1, \dots, l\}$  and its subgroup

$$\mathfrak{S}(l-1) := \{\sigma \in \mathfrak{S}(l) : \sigma(l) = l\}$$

as the set permutations of the same set  $[l]$  which have  $l$  as a fixpoint. Consider the set

$$\mathcal{X}_l = \{(\sigma_1, \sigma_2) \in \mathfrak{S}(l) \times \mathfrak{S}(l) :$$

the group generated by  $\sigma_1, \sigma_2$  acts transitively on the set  $[l]\}$ .

The group  $\mathfrak{S}(l-1)$  acts on  $\mathcal{X}_l$  by coordinate-wise conjugation:

$$\pi \cdot (\sigma_1, \sigma_2) := (\pi \sigma_1 \pi^{-1}, \pi \sigma_2 \pi^{-1}).$$

The orbits of this action define an equivalence relation  $\sim$  on  $\mathcal{X}_l$ ; the corresponding equivalence classes have a natural combinatorial interpretation as *non-labeled, rooted, bicolored, oriented maps with  $l$  edges* which is out of scope of the current paper (see [Śni15, Section 0.6] for details).

For a permutation  $\pi$  we denote by  $C(\pi)$  the set of its cycles.

We say that a triple  $(\sigma_1, \sigma_2, q)$  is an ‘*expander*’ [Śni15, Definition 9.2], see also [DFS10], if  $\sigma_1, \sigma_2 \in \mathfrak{S}(l)$  are permutations and  $q: C(\sigma_2) \rightarrow \{2, 3, \dots\}$  is a function on the set of cycles of  $\sigma_2$  with the following two properties:

$$\sum_{c \in C(\sigma_2)} q(c) = |C(\sigma_1)| + |C(\sigma_2)|$$

and for every set  $A \subset C(\sigma_2)$  such that  $A \neq \emptyset$  and  $A \neq C(\sigma_2)$  we have that

$$\#\{c \in C(\sigma_1) : c \text{ intersects at least one of the cycles in } A\} > \sum_{d \in A} [q(d) - 1].$$

The following is a refined version of the formula (3.13).

**Lemma 3.5** ([Śni15, Corollary 0.4]). *For each integer  $l \geq 1$  the top-degree part of the expansion of the character  $\text{Ch}_l$  as a polynomial in the variables  $\gamma, \mathcal{R}_2, \mathcal{R}_3, \dots$  is given by*

$$(3.14) \quad \text{Ch}_l = \sum_{[(\sigma_1, \sigma_2)] \in \mathcal{X}_l / \sim} \gamma^{l+1-|C(\sigma_1)|-|C(\sigma_2)|} \sum_{\substack{q: C(\sigma_2) \rightarrow \{2,3,\dots\} \\ (\sigma_1, \sigma_2, q) \text{ is an expander}}} \prod_{c \in C(\sigma_2)} \mathcal{R}_{q(c)} + \\ + (\text{terms of degree at most } l-1).$$

where the first sum runs over the representatives of the equivalence classes.

### 3.9. Proof of part (b) of Theorem 2.4.

3.9.1. *The scaling when  $\alpha$  is constant.* Part (b) of Theorem 2.4 concerns the trivial case  $\ell = 1$  which one can easily deduce from (3.13).

We shall concentrate on a specific case which will be useful in applications (see the proof of Theorem 1.9 in Section 3.10 below) and we shall assume that the refined asymptotics of characters specified in part (b) of Theorem 2.4 holds true for the quantity (2.6). For a specific choice of the constants

$$(3.15) \quad \chi_n(l) = a_{l+1} n^{-\frac{l-1}{2}} + b_{l+1} n^{-\frac{l}{2}} + o\left(n^{-\frac{l}{2}}\right) \quad \text{for } l \geq 1$$

for some sequences  $(a_l)$  and  $(b_l)$ , it is a simple exercise to use (2.1) and (3.13) in order to show that

$$(3.16) \quad \mathbb{E}_{\chi_n}(\text{Ch}_l) = a_{l+1} n^{\frac{l+1}{2}} + b_{l+1} n^{\frac{l}{2}} + o\left(n^{\frac{l}{2}}\right) \quad \text{for } l \geq 1,$$

$$(3.17) \quad \mathbb{E}_{\chi_n}(\mathcal{R}_{l+1}) = a'_{l+1} n^{\frac{l+1}{2}} + b'_{l+1} n^{\frac{l}{2}} + o\left(n^{\frac{l}{2}}\right) \quad \text{for } l \geq 1,$$

where  $(a'_l)$  is given again by (3.11) and  $(b'_l)$  is the unique sequence which fulfills

$$(3.18) \quad b_{l+1} = b'_{l+1} + \gamma \sum_{i \geq 1} \sum_{k_1 + \dots + k_i = l} \frac{l}{i} (k_1 - 1) \cdots (k_i - 1) a_{k_1} \cdots a_{k_i}.$$

In particular, (3.16) shows that the refined asymptotics of characters specified in part (b) of Theorem 2.4 holds true for the quantity (2.7).

Consider now the quantity under the limit symbol in (2.8) for  $\ell = 1$  and for the specific choice of  $x_1 = \mathcal{S}_l$ . Equation (3.4) implies that the refined asymptotics of characters specified in part (b) of Theorem 2.4 holds true for the quantity (2.8):

$$(3.19) \quad \mathbb{E}_{\chi_n}(\mathcal{S}_l) = a_l'' n^{\frac{l}{2}} + b_l'' n^{\frac{l-1}{2}} + o\left(n^{\frac{l-1}{2}}\right) \quad \text{for each } l \geq 2,$$

with the constants given by (3.12) and

$$b_l'' = \sum_{i \geq 1} \frac{1}{(i-1)!} (l-1)_{i-1} \sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = l}} b_{k_1}' a_{k_2}' \cdots a_{k_i}'.$$

We conclude the proof by pointing out that for  $\ell = 1$  the expression under the limit symbol in (2.8) coincides with its counterpart from (2.9).  $\square$

*Remark 3.6.* One can see that generically (even if the initial characters have small subleading terms which corresponds to  $b_l \equiv 0$ ) the subleading terms in (3.19) are much bigger than their counterparts from (3.10), namely they are of order  $\frac{1}{\sqrt{n}}$  times the leading asymptotic term. In particular, as we shall see in Section 3.10 below, the nonvanishing of  $(b_l'')$  and of the corresponding subleading term in (3.19) is the origin of the deterministic shift  $\mathbb{E}\Delta_\infty \neq 0$ .

**3.9.2. The double scaling limit.** In the double scaling limit considered in Theorem 1.13 the reasoning presented above remains valid if one replaces references to (3.13) by Lemma 3.5. Note, however, that the relationship (3.11) in this new context takes the form

$$a_{l+1} = \sum_{[(\sigma_1, \sigma_2)] \in \mathcal{X}_l / \sim} g^{l+1-|C(\sigma_1)|-|C(\sigma_2)|} \sum_{\substack{q: C(\sigma_2) \rightarrow \{2,3,\dots\} \\ (\sigma_1, \sigma_2, q) \text{ is an expander}}} \prod_{c \in C(\sigma_2)} a'_{q(c)},$$

while (3.18) takes the form

$$\begin{aligned} b_{l+1} = & \sum_{[(\sigma_1, \sigma_2)] \in \mathcal{X}_l / \sim} (l+1-|C(\sigma_1)|-|C(\sigma_2)|) g' g^{l-|C(\sigma_1)|-|C(\sigma_2)|} \times \\ & \times \sum_{\substack{q: C(\sigma_2) \rightarrow \{2,3,\dots\} \\ (\sigma_1, \sigma_2, q) \text{ is an expander}}} \prod_{c \in C(\sigma_2)} a'_{q(c)} + \\ & + \sum_{[(\sigma_1, \sigma_2)] \in \mathcal{X}_l / \sim} g^{l+1-|C(\sigma_1)|-|C(\sigma_2)|} \times \\ & \times \sum_{\substack{q: C(\sigma_2) \rightarrow \{2,3,\dots\} \\ (\sigma_1, \sigma_2, q) \text{ is an expander}}} \sum_{c \in C(\sigma_2)} b'_{q(c)} \prod_{c' \in C(\sigma_2) \setminus \{c\}} a'_{q(c')}. \end{aligned}$$



### 3.10. Proof of Theorem 1.9.

*Proof of Theorem 1.9.* From the very beginning we shall consider the more general setup of the double scaling limit from Theorem 1.13 in which the deformation parameter  $\alpha = \alpha(n)$  might depend on  $n$ .

The only difference with Theorem 1.7 is that instead of the family  $(X_k)$  of centered random variables, we deal now with the family of random variables  $(Y_k)$  given by (1.23) which are *not* centered. This means that the only remaining difficulty not covered by the proof of Theorem 1.7 presented in Section 2.7.2 is to show that the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}Y_k &= \lim_{n \rightarrow \infty} \sqrt{n} \left( \mathbb{E}\mathcal{S}_k^{(1)}(\Lambda_n) - \lim_{m \rightarrow \infty} \mathbb{E}\mathcal{S}_k^{(1)}(\Lambda_m) \right) = \\ &\quad \lim_{n \rightarrow \infty} \sqrt{n} \left( n^{-\frac{k}{2}} \mathbb{E}\mathcal{S}_k(\lambda_n) - \lim_{m \rightarrow \infty} m^{-\frac{k}{2}} \mathbb{E}\mathcal{S}_k(\lambda_m) \right) \end{aligned}$$

exists for each  $k \geq 2$ . A sufficient condition assuring the above limit exists is that the sequence  $\left( n^{-\frac{k}{2}} \mathbb{E}\mathcal{S}_k(\lambda_n) \right)_n$  is of the form (2.10).

By hypothesis, the enhanced approximate factorization property holds true (Definition 1.11) hence we can apply Theorem 2.4(b) which says that for each integer  $k \geq 2$  there exist constants  $a_k'', b_k''$  given explicitly in Section 3.9, such that

$$n^{-\frac{k}{2}} \mathbb{E}\mathcal{S}_k(\lambda_n) = a_k'' + \frac{b_k'' + o(1)}{\sqrt{n}}$$

as  $n \rightarrow \infty$  (see Equation (3.19)). In particular,

$$\lim_{n \rightarrow \infty} \mathbb{E}Y_k = b_k''$$

exists and is finite.

Similarly as in the proof of Theorem 1.7, we proved that the joint distribution of the random variables  $(Y_i)$  converges in moments as  $n \rightarrow \infty$  to the joint distribution of an abstract family of random variables  $(Z_i)$  with the property that all cumulants vanish:  $\kappa_\ell(Z_{i_1}, \dots, Z_{i_\ell}) = 0$ , except for  $\ell = 1, 2$ . The latter is the defining property of the (non-centered) Gaussian distribution. Since the Gaussian distribution is uniquely determined by its moments, it follows that  $(Y_i)$  converges to  $(Z_i)$  not only in moments but also in the weak topology of probability measures, as required.

The limiting object  $\Delta_\infty$ , the existence of which is provided by Theorem 1.9, is a random Schwartz distribution; its mean value  $\mathbb{E}\Delta_\infty$  is specified by test functions

$$\frac{k-1}{2} \int u^{k-2} \mathbb{E}\Delta_\infty(u) du = b_k''.$$

□

#### 4. MEASURES UNIQUELY DETERMINED BY THEIR MOMENTS

We shall present the second part of the proof of Theorem 1.6 (the first part of the proof was presented in Section 2.8.2). The remaining difficulty is to show that convergence of certain measures *in moments* implies also convergence *in the weak topology of probability measures*. We are going to do it by showing that the sequence of moments uniquely determines the corresponding measure.

##### 4.1. Slowly growing sequence of moments determines the measure.

**Lemma 4.1.** *Assume that  $\mu$  is a probability measure which is supported on the interval  $[x_0, \infty)$  (respectively, on the interval  $(-\infty, x_0]$ ) for some  $x_0 \in \mathbb{R}$  and such that*

$$(4.1) \quad |m_l| \leq C^l l^{2l}$$

*holds true for some constant  $C$  and all integers  $l \geq 1$ , where*

$$m_l = m_l(\mu) = \int_{\mathbb{R}} x^l d\mu$$

*is the  $l$ -th moment of  $\mu$ .*

*Then the measure  $\mu$  is uniquely determined by its moments.*

*Similarly, if the measure  $\mu$  is supported on the real line  $\mathbb{R}$  and such that*

$$(4.2) \quad |m_l| \leq C^l l^l$$

*holds true for some constant  $C$  and all integers  $l \geq 1$ , then the measure  $\mu$  is uniquely determined by its moments.*

*Proof.* In the case when  $\mu$  is supported on the interval  $[0, \infty)$  this is exactly Stieltjes moment problem, while in the case when  $\mu$  is supported on the real line  $\mathbb{R}$  this is exactly the Hamburger moment problem. It is easy to check that the assumptions (4.1), and (4.2) imply that the Carleman's conditions in both Stieltjes and Hamburger, respectively, problems are satisfied and it follows that the measure  $\mu$  is uniquely determined by its moments.

Now, assume that  $\mu$  is a probability measure which is supported on the interval  $[x_0, \infty)$ . We define a probability measure  $\mu_{x_0}$  supported on the interval  $[0, \infty)$ , as a translation of  $\mu$  that is, for any measurable set  $A \subset \mathbb{R}$  we have

$$\mu_{x_0}(A) := \mu(A + x_0).$$

Let us compute the moments of  $\mu_{x_0}$ :

$$m_l(\mu_{x_0}) = \int_{\mathbb{R}} x^l d\mu_{x_0} = \int_{\mathbb{R}} (x - x_0)^l d\mu = \sum_k \binom{l}{k} (-x_0)^{l-k} \int_{\mathbb{R}} x^k d\mu.$$

This leads to the following inequalities:

$$\begin{aligned} |m_l(\mu_{x_0})| &\leq \sum_k \binom{l}{k} |x_0|^{l-k} |m_k| \leq \\ &\sum_k \binom{l}{k} |x_0|^{l-k} C^k k^{2k} \leq (C + |x_0|)^l l^{2l}. \end{aligned}$$

By Carleman's criterion it means that the measure  $\mu_{x_0}$  is uniquely determined by its moments, which is equivalent by the construction that the measure  $\mu$  is uniquely determined by its moments, too.

The case, when  $\mu$  is supported on the interval  $(-\infty, x_0]$  is analogous, and we leave it as a simple exercise.  $\square$

#### 4.2. Slow growth of $(\mathcal{R}_n)$ implies slow growth of $(\mathcal{S}_n)$ .

**Lemma 4.2.** *Let  $(\mathcal{S}_l)_{l \geq 2}$  be a sequence of real numbers and let  $(\mathcal{R}_l)_{l \geq 2}$  given by (3.5) be the corresponding sequence of free cumulants. Assume that the sequence of free cumulants fulfills the estimate*

$$(4.3) \quad |\mathcal{R}_l| \leq C^l l^{ml}$$

for some constants  $m, C \geq 0$  and all  $l \geq 2$ .

Then the sequence  $(\mathcal{S}_l)$  fulfills an analogous estimate

$$(4.4) \quad |\mathcal{S}_l| \leq C^l l^{ml};$$

possibly for another value of the constant  $C$ .

*Proof.* The expansion (3.4) for  $\mathcal{S}_l$  in terms of the free cumulants gives immediately:

$$\begin{aligned} (4.5) \quad |\mathcal{S}_l| &\leq \sum_{i \geq 1} \frac{1}{i!} (l-1)_{i-1} C^l \sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = l}} k_1^{mk_1} \dots k_i^{mk_i} \\ &\leq C^l l^{ml} \sum_{i \geq 1} \frac{1}{i!} (l-1)_{i-1} \sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = l}} 1. \end{aligned}$$

Since

$$\sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = l}} 1 = \sum_{\substack{k_1, \dots, k_i \geq 1 \\ k_1 + \dots + k_i = l-i}} 1 = \binom{l-1-i}{i-1},$$

we can bound the sum on the right-hand side of (4.5) as follows:

$$\begin{aligned} \sum_{i \geq 1} \frac{1}{i!} (l-1)_{i-1} \sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = l}} 1 &\leq \sum_{i \geq 1} \binom{l-1}{i-1} \binom{l-1-i}{i-1} \leq \\ &\sum_{i \geq 0} \binom{l-1}{i}^2 \leq \left( \sum_{i \geq 0} \binom{l-1}{i} \right)^2 \leq 2^{2l}, \end{aligned}$$

which plugged into (4.5) yields

$$|\mathcal{S}_l| \leq (4C)^l l^{ml},$$

which finishes the proof.  $\square$

**4.3. Estimates on some classes of permutations.** Recall that for a permutation  $\pi$  we denote by  $C(\pi)$  the set of its cycles. The length

$$\|\pi\| := l - |C(\pi)|$$

of a permutation  $\pi \in \mathfrak{S}(l)$  is defined as the minimal number of factors necessary to write  $\pi$  as a product of transpositions.

**Lemma 4.3.** *For all integers  $r \geq 0$  and  $l \geq 1$*

$$\#\left\{ \pi \in \mathfrak{S}(l) : \|\pi\| = r \right\} \leq \frac{l^{2r}}{r!}.$$

*Proof.* We claim that for each permutation  $\pi \in \mathfrak{S}(l)$  such that  $\|\pi\| = r$  there exist at least  $r$  transpositions  $\tau$  with the property that the permutation  $\pi' := \pi\tau$  fulfills  $\|\pi'\| = \|\pi\| - 1$ . Indeed, each such a transposition is of the form  $\tau = (a, b)$  with  $a \neq b$  being elements of the same cycle of  $\pi$ ; it follows that the number of such transpositions is equal to

$$\sum_{c \in C(\pi)} \binom{|c|}{2} \geq \sum_{c \in C(\pi)} (|c| - 1) = \|\pi\|.$$

By repeating inductively the same argument for the collection of permutations  $(\pi')$  obtained above, it follows that the permutation  $\pi$  can be written in at least  $r!$  different ways as a product of  $r$  transpositions. Since there are  $\binom{l}{2} < l^2$  transpositions in  $\mathfrak{S}(l)$ , this concludes the proof.  $\square$

We revisit Section 3.8.2. In the following we will need a convenient way of parametrizing the equivalence classes in  $\mathcal{X}_l / \sim$ ; for this purpose we note that in each equivalence class one can choose a representative (which is not necessarily unique)  $(\sigma_1, \sigma_2)$  with the property that the permutation  $\sigma_2$  has a particularly simple cycle structure, namely

$$\sigma_2 = (1, 2, \dots, i_1)(i_1 + 1, i_1 + 2, \dots, i_2) \cdots (i_{\ell-1} + 1, i_{\ell-1} + 2, \dots, i_\ell)$$

for some increasing sequence  $1 \leq i_1 < i_2 < \dots < i_\ell = l$ . Note that for a fixed  $\ell = |C(\sigma_2)|$

(4.6) the number of permutations  $\sigma_2$  of the above form

$$\text{is given by } \binom{l}{\ell-1} \leq l^{\ell-1}.$$

#### 4.4. Growth of free cumulants.

**Proposition 4.4.** *Consider the double scaling limit in which  $\alpha = \alpha(n)$  depends on  $n$  as in Theorem 1.13. Assume that we are given a sequence  $(\chi_n)$  of reducible Jack characters  $\chi_n: \mathcal{P}_n \rightarrow \mathbb{R}$  for which Hypothesis 2.5 holds true. We define*

$$(4.7) \quad r_l := \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{l}{2}}} \mathbb{E}_{\chi_n} \mathcal{R}_l.$$

Then

$$(4.8) \quad |r_l| \leq C^l l^{ml}$$

holds true for some constant  $C$  and each integer  $l \geq 2$ , where  $m$  is given by (2.16).

*Proof.* Approximate factorization property, by Theorem 2.4, implies that the limit (4.7) exists and is finite for each integer  $l \geq 2$  and, furthermore, the expected value of a product of free cumulants approximately factorizes:

$$(4.9) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{k_1 + \dots + k_\ell}{2}}} \mathbb{E}_{\chi_n} [\mathcal{R}_{k_1} \cdots \mathcal{R}_{k_\ell}] = r_{k_1} \cdots r_{k_\ell}.$$

Let us consider first the case  $g = 0$ . Let us divide both sides of (3.13) by  $n^{\frac{l+1}{2}}$  and take the mean value  $\mathbb{E}_{\chi_n}$ . By taking the limit  $n \rightarrow \infty$  and using (4.9) we obtain in this way

$$r_{l+1} = a_{l+1}$$

and the claim follows immediately.

From the following on we consider the generic case  $g \neq 0$  and  $m = 2$ . Analogously as above, let us divide both sides of (3.14) by  $n^{\frac{l+1}{2}}$  and take the mean value  $\mathbb{E}_{\chi_n}$ . By taking the limit  $n \rightarrow \infty$  and using (4.9) we obtain in this way

$$(4.10) \quad a_{l+1} = \sum_{[(\sigma_1, \sigma_2)] \in \mathcal{X}_l / \sim} g^{l+1-|C(\sigma_1)|-|C(\sigma_2)|} \sum_{\substack{q: C(\sigma_2) \rightarrow \{2,3,\dots\} \\ (\sigma_1, \sigma_2, q) \text{ is an expander}}} \prod_{c \in C(\sigma_2)} r_{q(c)}.$$

We shall cluster the summands according to the parameters  $u$  and  $v$  given by

$$u := \|\sigma_1\| \quad \text{and} \quad v := l + 1 - |C(\sigma_1)| - |C(\sigma_2)|.$$

In the following we will show that from the transitivity requirement in the definition of  $\mathcal{X}_l$  it follows that

$$v \geq 0.$$

Indeed, let us construct a bipartite graph  $G = (V_\circ \sqcup V_\bullet, E)$  with the vertices corresponding to the cycles of  $\sigma_1$  and the cycles of  $\sigma_2$ :  $V_\circ = C(\sigma_1)$ ,  $V_\bullet = C(\sigma_2)$ ; we connect two vertices by an edge if the corresponding cycles are not disjoint, that is  $e = (c_1, c_2) \in E$  if  $c_1 \cap c_2 \neq \emptyset$ . Then, the transitivity of the action of the group generated by  $\sigma_1, \sigma_2 \in \mathfrak{S}(l)$  means precisely that the graph  $G$  is connected. Since the number of edges of  $G$  is bounded from above by  $l$  it follows that the number of vertices of  $G$ , which is equal to  $|C(\sigma_1)| + |C(\sigma_2)|$ , cannot exceed  $l + 1$ , which gives the required inequality.

Furthermore, the contribution of the terms for which the equality  $v = 0$  holds true corresponds to the specialization  $g = 0$ ; by revisiting (3.13) it follows that this contribution is equal to  $r_{l+1}$  which corresponds to the unique equivalence class

$$\{(\text{id}, \sigma_2) : \sigma_2 \in \mathfrak{S}(l) \text{ is such that } |C(\sigma_2)| = 1\}$$

for which  $u = 0$  and  $v = 0$ . It is easy to check that it is the unique summand for which  $u = 0$  (since the latter condition is equivalent to  $\sigma_1 = \text{id}$ ). By singling out this particular summand, (4.10) can be transformed to

$$(4.11) \quad r_{l+1} = a_{l+1} - \sum_{\substack{u \geq 1 \\ v \geq 1}} \sum_{\substack{[(\sigma_1, \sigma_2)] \in \mathcal{X}_l / \sim \\ \|\sigma_1\| = u \\ l+1-|C(\sigma_1)|-|C(\sigma_2)| = v}} g^v \sum_{\substack{q: C(\sigma_2) \rightarrow \{2,3,\dots\} \\ (\sigma_1, \sigma_2, q) \text{ is an expander}}} \prod_{c \in C(\sigma_2)} r_{q(c)}.$$

With the notations used in Equation (4.11)

$$(4.12) \quad v + \sum_{c \in C(\sigma_2)} q(c) = v + |C(\sigma_1)| + |C(\sigma_2)| = l + 1$$

which implies that for each  $C > 0$  we have that

$$(4.13) \quad \left| \frac{r_{l+1}}{C^{l+1} (l+1)^{2(l+1)}} \right| \leq \left| \frac{a_{l+1}}{C^{l+1} (l+1)^{2(l+1)}} \right| + \sum_{\substack{u \geq 1 \\ v \geq 1}} \sum_{\substack{[(\sigma_1, \sigma_2)] \in \mathcal{X}_l / \sim \\ \|\sigma_1\| = u \\ l+1 - |C(\sigma_1)| - |C(\sigma_2)| = v}} \left( \frac{|g|}{C (l+1)^2} \right)^v \times \\ \times \sum_{\substack{q: C(\sigma_2) \rightarrow \{2, 3, \dots\} \\ (\sigma_1, \sigma_2, q) \text{ is an expander}}} \prod_{c \in C(\sigma_2)} \left| \frac{r_{q(c)}}{C^{q(c)} (l+1)^{2q(c)}} \right|.$$

By Lemma 4.3 and (4.6), for each pair of integers  $u, v \geq 1$  the number of equivalence classes  $[(\sigma_1, \sigma_2)]$  which could possibly contribute to the above sum is bounded from above by

$$\frac{l^{2u}}{u!} (l+1)^{|C(\sigma_2)|-1} = \frac{l^{2u}}{u!} (l+1)^{u-v} \leq \frac{(l+1)^{3u-v}}{u!}.$$

For each representative  $(\sigma_1, \sigma_2)$  of an equivalence class the number of functions  $q: C(\sigma_2) \rightarrow \{2, 3, \dots\}$  which fulfill (4.12) is bounded from above by

$$(l+1)^{|C(\sigma_2)|-1} = (l+1)^{u-v}.$$

Our strategy is to prove (4.8) for  $m = 2$  by induction over  $l \geq 2$ . It is trivial to check that  $r_2 = 1$  always holds true and thus the induction base  $l = 2$  is valid if  $C \geq 1$ . We shall assume that (4.8) holds true for  $2 \leq k \leq l$ . The value of the constant  $C$  will be specified at the end of the proof in such a way that each induction step can be justified. The induction hypothesis implies that

$$\prod_{c \in C(\sigma_2)} |r_{q(c)}| \leq C^{l+1-v} \prod_{c \in C(\sigma_2)} q(c)^{2q(c)}.$$

We claim that the following inequality holds true:

$$(4.14) \quad \prod_{c \in C(\sigma_2)} q(c)^{2q(c)} \leq 2^{4(u-v)} (l+1)^{2(l+1-v-2(u-v))}.$$

Indeed, the logarithm of the left-hand side is a convex function

$$\mathbb{R}_+^{|C(\sigma_2)|} \ni (q(c) : c \in C(\sigma_2)) \mapsto 2 \sum_{c \in C(\sigma_2)} q(c) \log q(c);$$

its supremum over the simplex given by inequalities  $q(c) \geq 2$  and the equality (4.12) is attained in one of the simplex vertices which corresponds to

$$(q(c))_{c \in C(\sigma_2)} = (l+1-v-2(u-v), \underbrace{2, \dots, 2}_{|C(\sigma_2)|-1=u-v \text{ times}});$$

this concludes the proof of (4.14).

In this way we proved that

$$\prod_{c \in C(\sigma_2)} \left| \frac{r_{q(c)}}{C^{q(c)}(l+1)^{2q(c)}} \right| \leq \left( \frac{2}{l+1} \right)^{4(u-v)}.$$

It follows that the right-hand side of (4.13) is bounded from above by

$$\begin{aligned} & \left| \frac{a_{l+1}}{C^{l+1} (l+1)^{2(l+1)}} \right| + \sum_{\substack{u \geq 1 \\ v \geq 1}} \frac{(l+1)^{3u-v}}{u!} (l+1)^{u-v} \left( \frac{|g|}{C(l+1)^2} \right)^v \left( \frac{2}{l+1} \right)^{4(u-v)} \leq \\ & \left| \frac{a_{l+1}}{C^{l+1} (l+1)^{2(l+1)}} \right| + \sum_{u \geq 1} \frac{2^{4u}}{u!} \sum_{v \geq 1} \left( \frac{|g|}{2^{4C}} \right)^v = \\ & \left| \frac{a_{l+1}}{C^{l+1} (l+1)^{2(l+1)}} \right| + (e^{16} - 1) \frac{\frac{|g|}{2^{4C}}}{1 - \frac{|g|}{2^{4C}}} \end{aligned}$$

for  $\frac{|g|}{2^{4C}} < 1$ . The right-hand side tends to zero uniformly over  $l$  as  $C \rightarrow \infty$ ; there exists therefore some  $C$  such that the right-hand side is smaller than 1. Such a choice of  $C$  assures that each inductive step is justified. This concludes the proof.  $\square$

#### 4.5. The second part of the proof of Theorem 1.6.

*The second part of the proof of Theorem 1.6.* We continue the discussion from Section 2.8.2. The only missing component of the proof is to show that the probability measure  $P_{\Lambda_\infty}$  is uniquely determined by its moments. We recall equation (2.17) which says that its moments

$$\int u^k dP_{\Lambda_\infty}(u) = \frac{1}{k+1} s_{k+2}$$

are related to the fundamental functionals of shape by the following formula

$$s_k := \lim_{n \rightarrow \infty} \mathbb{E}_{\chi_n} \frac{1}{n^{\frac{k}{2}}} \mathcal{S}_k.$$



The proof of (4.9) can be adapted to the functionals of shape, therefore

$$(4.15) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{k_1 + \dots + k_\ell}{2}}} \mathbb{E}_{\chi_n} [\mathcal{R}_{k_1} \cdots \mathcal{R}_{k_\ell}] = r_{k_1} \cdots r_{k_\ell}.$$

Since fundamental functionals of shape and free cumulants are related by equation (3.5), by (4.15) it follows that the relation between the families of real numbers  $(s_k)_{k \geq 2}$  and  $(r_k)_{k \geq 2}$  is given by an analogue of (3.5) as well, where

$$r_k := \lim_{n \rightarrow \infty} \mathbb{E}_{\chi_n} \frac{1}{n^{\frac{k}{2}}} \mathcal{R}_k,$$

cf. (4.7).

Proposition 4.4 states that there exists some constant  $C$  such that

$$|r_k| \leq C^k k^{mk}$$

for all positive integers  $k \geq 2$ , where  $m$  is given by (2.16). Thus Lemma 4.2 gives us the following estimates for the moments of  $P_{\Lambda_\infty}$ :

$$\left| \int u^k dP_{\Lambda_\infty}(u) \right| = \left| \frac{1}{k+1} s_{k+2} \right| \leq C'^{k+2} (k+2)^{m(k+2)} \leq C''^k k^{mk}$$

for some constants  $C', C''$ .

We consider first the case  $g = 0$ . Lemma 4.1 implies immediately that the measure  $P_{\Lambda_\infty}$  is uniquely determined by its moments which concludes the proof.

In the case  $g > 0$  the height of each box constituting the anisotropic Young diagram  $\Lambda_n$  is equal to  $g + o(1) > c$  for some constant  $c > 0$ , uniformly over  $n$ , cf. Section 1.12. By comparison of the areas it follows that the length  $l$  of the bottom rectangle constituting  $\Lambda_n$  fulfills  $lc \leq 1$ ; in particular it follows that the support of the measure  $P_{\Lambda_n}$  is contained in the interval  $(-\infty, \frac{1}{c}]$ . It follows that an analogous inclusion holds true for the support of the mean value  $\mathbb{E}P_{\Lambda_n}$ ; by passing to the limit the same is true for  $P_{\Lambda_\infty}$ . It follows that Lemma 4.1 can be applied which concludes the proof.

The case  $g < 0$  is fully analogous. □

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## REFERENCES

- [BGG16] A. Borodin, V. Gorin, and A. Guionnet. Gaussian asymptotics of discrete  $\beta$ -ensembles. *Publ.math.IHES (to appear)*, 2016. arXiv:1505.03760.
- [Bia98] P. Biane. Representations of symmetric groups and free probability. *Adv. Math.*, 138(1):126–181, 1998.
- [Bia01] P. Biane. Approximate factorization and concentration for characters of symmetric groups. *Internat. Math. Res. Notices*, (4):179–192, 2001.
- [DF16] M. Dołęga and V. Féray. Gaussian fluctuations of Young diagrams and structure constants of Jack characters. *Duke Math. J.*, 165(7):1193–1282, 2016. doi: 10.1215/00127094-3449566.
- [DFŚ10] M. Dołęga, V. Féray, and P. Śniady. Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations. *Adv. Math.*, 225(1):81–120, 2010.
- [DŚ17] Maciej Dołęga and Piotr Śniady. Examples of Jack-deformed random Young diagrams. In preparation, 2017.
- [For10] P. J. Forrester. *Log-gases and random matrices*, volume 34 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2010.
- [FRT54] J.S. Frame, G. de B. Robinson, and R. M. Thrall. The hook graphs of the symmetric group. *Canad. J. Math.*, 6(316):C324, 1954.
- [Ful04] J. Fulman. Stein’s method, Jack measure, and the Metropolis algorithm. *Journal of Combinatorial Theory, Series A*, 108(2):275–296, 2004.
- [Jac71] H. Jack. A class of symmetric polynomials with a parameter. *Proc. Roy. Soc. Edinburgh Sect. A*, 69:1–18, 1970/1971.
- [Ker93a] Sergei Kerov. Gaussian limit for the Plancherel measure of the symmetric group. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(4):303–308, 1993.
- [Ker93b] Sergei Kerov. Transition probabilities of continual Young diagrams and the Markov moment problem. *Funct. Anal. Appl.*, 27(3):104–117, 1993.
- [Ker96] S. Kerov. The boundary of Young lattice and random Young tableaux. In *Formal power series and algebraic combinatorics (New Brunswick, NJ, 1994)*, volume 24 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 133–158. Amer. Math. Soc., Providence, RI, 1996.
- [Ker00] Sergei Kerov. Anisotropic Young diagrams and Jack symmetric functions. *Funct. Anal. Appl.*, 34:41–51, 2000.
- [KO94] Sergei Kerov and Grigori Olshanski. Polynomial functions on the set of Young diagrams. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(2):121–126, 1994.
- [KOO98] S. Kerov, A. Okounkov, and G. Olshanski. The boundary of the young graph with jack edge multiplicities. *International Mathematics Research Notices*, 1998(4):173–199, 1998.
- [Las08] M. Lassalle. A positivity conjecture for Jack polynomials. *Math. Res. Lett.*, 15(4):661–681, 2008.
- [Las09] M. Lassalle. Jack polynomials and free cumulants. *Adv. Math.*, 222(6):2227–2269, 2009.
- [LS77] B. F. Logan and L. A. Shepp. A variational problem for random Young tableaux. *Advances in Math.*, 26(2):206–222, 1977.
- [Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York,

- second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [Mat08] S. Matsumoto. Jack deformations of Plancherel measures and traceless Gaussian random matrices. *Elec. Jour. Comb.*, 15(R149):1, 2008.
  - [Mél11] P.-L. Méliot. Kerov’s central limit theorem for Schur–Weyl and Gelfand measures. In *23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, Discrete Math. Theor. Comput. Sci. Proc., AO, pages 669–680. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011.
  - [Mol15] Alexander Moll. Random partitions and the quantum Benjamin–Ono hierarchy. Preprint arXiv:1508.03063, 2015.
  - [Oko03] A. Okounkov. The uses of random partitions. In *Fourteenth International Congress on Mathematical Physics*, pages 379–403. World Scientists, 2003.
  - [Sag01] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.
  - [Śni06a] Piotr Śniady. Gaussian fluctuations of characters of symmetric groups and of Young diagrams. *Probab. Theory Related Fields*, 136(2):263–297, 2006.
  - [Śni06b] Piotr Śniady. Gaussian fluctuations of characters of symmetric groups and of Young diagrams. *Probab. Theory Related Fields*, 136(2):263–297, 2006.
  - [Śni15] P. Śniady. Top degree of Jack characters and enumeration of maps. Preprint arXiv:1506.06361, 2015.
  - [Śni16] Piotr Śniady. Structure coefficients for Jack characters: approximate factorization property. Preprint arXiv:1603.04268, 2016.
  - [Sta89] R. P. Stanley. Some combinatorial properties of Jack symmetric functions. *Adv. Math.*, 77(1):76–115, 1989.
  - [Tho64] Elmar Thoma. Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. *Math. Z.*, 85:40–61, 1964.
  - [Ver95] Anatoly M. Vershik. Asymptotic combinatorics and algebraic analysis. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1384–1394. Birkhäuser, Basel, 1995.
  - [VK77] A. M. Vershik and S. V. Kerov. Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux. *Dokl. Akad. Nauk SSSR*, 233(6):1024–1027, 1977.
  - [VK81] A. M. Vershik and S. V. Kerov. Asymptotic theory of the characters of a symmetric group. *Funktsional. Anal. i Prilozhen.*, 15(4):15–27, 96, 1981.

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